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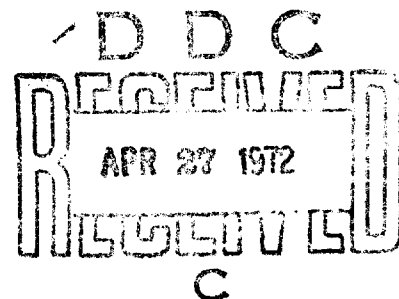# FORMULATION OF THE LARGE DEFLECTION SHELL EQUATIONS FOR USE IN FINITE DIFFERENCE STRUCTURAL RESPONSE COMPUTER CODES

by

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February 1972

U.S. ARMY ABERDEEN RESEARCH AND DEVELOPMENT CENTER
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	ROLE	WT	ROLE	WT	ROLE	WT
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Pressure Loading						
Elastoplastic Material Behavior						
Strain Hardening Behavior						
Strain Rate Sensitive Behavior						
Computer Program						

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LIST OF SYMBOLS*

a	determinant of $a_{\alpha\beta}$
$\underset{\sim}{a}_\alpha$	basis for reference surface
$\underset{\sim}{a}^\alpha$	dual of the basis $\underset{\sim}{a}_\alpha$
$a_{\alpha\beta}, a^{\alpha\beta}$	covariant and contravariant components of the metric (first fundamental tensor) of the reference surface
$\underset{\sim}{b}$	body force vector
b^i	components of $\underset{\sim}{b}$ in the basis $\underset{\sim}{g}_i$
$b_{\alpha\beta}, b^\alpha_\beta$	covariant and mixed components of the second fundamental tensor of the reference surface
$\underset{\sim}{c}$	external couple per unit area of reference surface
$\underset{\sim}{C}$	tangential vector equivalent to $\underset{\sim}{c}$, $\underset{\sim}{c} \times \underset{\sim}{n}$
C^α	components of $\underset{\sim}{C}$ in the basis $\underset{\sim}{a}_\alpha$
$C^{*\alpha}$	$a^{\frac{1}{2}} C^\alpha$
$\underset{\sim}{\hat{C}}^\alpha$	normal vectors of magnitudes C^α, vis. $C^\alpha \underset{\sim}{n}$
E	Young's modulus
$E_{\alpha\beta}$	tangential components of strain tensor in the basis $\underset{\sim}{g}_\alpha$
$\underset{\sim}{F}$	external force per unit area of reference surface
$\underset{\sim}{F}^*$	$a^{\frac{1}{2}} \underset{\sim}{F}$
F^α, F	tangential and normal components of $\underset{\sim}{F}$ in basis $(\underset{\sim}{a}_\alpha, \underset{\sim}{n})$
g	determinant of g_{ij} or equivalently $g_{\alpha\beta}$
$\underset{\sim}{g}_i$	basis in Euclidean space
$\underset{\sim}{g}^i$	dual of the basis $\underset{\sim}{g}_i$
g_{ij}, g^{ij}	covariant and contravariant components of the metric in Euclidean space

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*Superscripts and subscripts range over the integers as follows: latin 1,2,3; greek 1,2; script 0,1,2.

h thickness of shell

$\underset{\sim}{\ell}$ spin momentum per unit area of reference surface

$\underset{\sim}{m}_{(\nu)}$ couple resultant vector per unit arc length, $\underset{\sim}{m}^{\alpha}\nu_{\alpha}$

$\underset{\sim}{m}^{\alpha}$ couple resultant tensor

$\underset{\sim}{M}^{\alpha}$ bending resultant tensor, $\underset{\sim}{m}^{\alpha} \times \underset{\sim}{n}$

$M^{\alpha\beta}$ bending components; $M^{\alpha\beta} = \underset{\sim}{M}^{\alpha}\cdot\underset{\sim}{a}^{\beta}$

$\hat{M}^{\alpha\beta}$ symmetric part of $M^{\alpha\beta}$

$\hat{\underset{\sim}{M}}^{\alpha\beta}$ normal vectors of magnitudes $\hat{M}^{\alpha\beta}$, vis. $\hat{M}^{\alpha\beta}\underset{\sim}{n}$

$\hat{\underset{\sim}{M}}{*}^{\alpha\beta}$ $a^{\frac{1}{2}}\hat{M}^{\alpha\beta}$

$\hat{\underset{\sim}{M}}{*}^{\alpha\beta}$ normal vectors of magnitudes $\hat{M}{*}^{\alpha\beta}$, vis. $\hat{M}{*}^{\alpha\beta}\underset{\sim}{n}$

$\underset{\sim}{n}$ unit normal to reference surface

$\hat{\underset{\sim}{N}}{*}^{\alpha}$ modified stress resultant tensor defined by (5.46)

$\underset{\sim}{P}$ linear momentum per unit area of reference surface

$\underset{\sim}{Q}_{(\nu)}$ stress resultant vector per unit arc length, $\underset{\sim}{Q}^{\alpha}\nu_{\alpha}$

$\underset{\sim}{Q}^{\alpha}$ stress resultant tensor

$\hat{\underset{\sim}{Q}}^{\alpha}$ modified stress resultant tensor defined by $(5.40)_2$

Q^{α} shear components of $\underset{\sim}{Q}^{\alpha}$; $Q^{\alpha} = \underset{\sim}{Q}^{\alpha}\cdot\underset{\sim}{n}$

$Q^{\alpha\beta}$ membrane components of $\underset{\sim}{Q}^{\alpha}$; $Q^{\alpha\beta} = \underset{\sim}{Q}^{\alpha}\cdot\underset{\sim}{a}^{\beta}$

$\hat{Q}^{\alpha\beta}$ modified membrane components defined by (5.35)

$\hat{Q}{*}^{\alpha\beta}$ $a^{\frac{1}{2}}\hat{Q}^{\alpha\beta}$

$\underset{\sim}{r}$ position vector of particles on reference surface

t time

$\underset{\sim}{t}_{+}, \underset{\sim}{t}_{-}$ traction on bounding surfaces S_{+} and S_{-} of shell

t^{i}_{+}, t^{i}_{-} components of $\underset{\sim}{t}_{+}$ and $\underset{\sim}{t}_{-}$ in the basis $\underset{\sim}{g}_{i}$

LIST OF SYMBOLS   (Continued)

$\Delta \underset{\sim}{u}$ displacement increment undergone by particles on the reference surface in the time interval Δt

$\underset{\sim}{v}$ velocity of particles on the reference surface

v^{α}, v tangential and normal components of $\underset{\sim}{v}$ in basis $(\underset{\sim}{a}_{\alpha}, \underset{\sim}{n})$

$\nabla \cdot \underset{\sim}{v}$ tangential divergence of $\underset{\sim}{v}$ as defined by (2.27)

$\underset{\sim}{x}$ position vector of shell particles, $\underset{\sim}{r} + \zeta \underset{\sim}{n}$

γ_{α} resultant moments of mass density defined by (4.15)

$\gamma^{*}{}_{\alpha}$ $a^{\frac{1}{2}} \gamma_{\alpha}$

$\Gamma_{\alpha\beta}{}^{\gamma}$ components of the connection of the reference surface, equal to the Christoffel symbols

δ^{β}_{α} Kronecker delta

$\varepsilon_{\alpha\beta}, \varepsilon^{\alpha\beta}$ covariant and contravariant components of the skew-symmetric tensor $\underset{\sim}{\varepsilon}$ of the reference surface

ζ distance of particle from reference surface along $\underset{\sim}{n}$

μ determinant of μ^{α}_{β}

μ^{α}_{β} components of tensor relating $\underset{\sim}{g}_{\alpha}$ to $\underset{\sim}{a}_{\alpha}$, see $(2.37)_1$

ν Poisson's ratio

$\underset{\sim}{\nu}$ exterior unit normal to curve in reference surface, defined in Appendix A

$\nu_{\alpha}, \nu^{\alpha}$ covariant and contravariant components of $\underset{\sim}{\nu}$

ξ^{α} material coordinates of particle on reference surface

ρ mass density per unit volume

$\underset{\sim}{\sigma}$ stress tensor

σ_{0} uniaxial yield stress

σ^{ij} contravariant components of $\underset{\sim}{\sigma}$ in the basis $\underset{\sim}{g}_{i}$

$\underset{\sim}{\tau}$ unit tangent to curve in reference surface, defined in Appendix A

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τ_α, τ^α covariant and contravariant components of $\underset{\sim}{\tau}$

Superscripts

$[\alpha\beta]$ antisymmetric part with respect to the α and β indices, see footnote p. 66

$(\alpha\beta)$ symmetric part with respect to the α and β indices, see footnote p. 66

$+,-$ values of variable at beginning and end of time interval Δt, cf. Chapter 3 and 7

$(+),(-)$ one sided limits of variable along arc, used in Chapter 6

Subscripts

$'\alpha$ partial derivative with respect to ξ^α

$|\alpha$ covariant derivative with respect to ξ^α

$+,-$ values of variable adjacent to symmetry boundary, employed in Chapter 6 and Appendix B only

I. INTRODUCTION

This report is mainly concerned with presenting a formulation of the equations governing the large transient deformaticns of shells, especially suitable for use in finite difference computer codes. The recent adaptation of these equations to REPSIL, a code developed at the BRL for predicting the large deflections of thin shells subjected to blast and impulse loadings, has resulted in a number of improvements, which are also described in this report. While the equations as formulated in the main body of the report are quite general, being restricted only by the Kirchhoff hypothesis, as applied in REPSIL they are further restricted to thin elastoplastic shells for which the rotatory inertia is negligible. A detailed description of the REPSIL code will be included in a forthcoming user's manual.

1.1 Historic Background

In order to put this report and the associated computer code REPSIL in their proper relation to work done at the Aeroelastic and Structures Research Laboratory of MIT on similar codes, a short history cf the development of REPSIL is in order. REPSIL has evolved from a computer code developed by LEECH [1]* originally called PETROS and now called PETROS 1 since the recent development by MORINO, LEECH and WITMER [2] of the improved PETROS 2. For its time PETROS 1 was the most sophisticated code available treating the large transient responce of shells. For this reason it was one of the codes selected for use in the numerical analysis portion of the research program on shells and structures at the BRL.

Upon close study, HUFFINGTON [3] found that PETROS 1 contained a number of flaws, in addition to having a rather limited choice of printing options. He discovered that the clamped edge boundary condition

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*Numbers in brackets refer to the list of references on page 123.

was incorrectly formulated and that the linear approximation employed in the stress increment calculation during plastic flow gave stress states outside the yield surface. He revised the code, by increasing the printing options, adding plotting capability, correcting the clamped edge condition and reformulating the stress calculation based on the exact quadratic expression. The revised code was named REPSIL. The details of the revised stress calculation are documented in [4]*.

In this early version, however, REPSIL still retained many of the unsatisfactory features of PETROS 1. For example, only cylindrical panels with clamped edges could be treated, the finite difference mesh had to be square, the strain-displacement relation did not properly account for the curvature of the shell, the finite difference derivatives along the boundaries were inconsistent, etc.** Consequently, a complete reformulation of REPSIL, including its theoretical foundation, was begun at the BRL. The reformulation has advanced sufficiently to warrant documentation of the present version of REPSIL. This report documents the mathematical model on which REPSIL is presently based. A user's manual documenting the computational reformulation of REPSIL will follow shortly.

The Aeroelastic and Structures Research Laboratory of MIT has also undertaken a complete revision of PETROS 1, which has resulted in the code PETROS 2 [2]. Although REPSIL and PETROS 2 share a number of features, they are independent extentions of PETROS 1 and do not coincide. The differences between REPSIL and PETROS 2, as well as their differences from PETROS 1, are discussed in Chapter 8.

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* PETROS 2 [2] uses a slightly modified version of this stress calculation.

** My colleague, Dr. Huffington, was responsible for first making me aware of some of these flaws.

1.2 Outline of Remainder of Report

The next five chapters, Chapters 2 through 6, develop rigorously the theory of the motion of shells subject to the Kirchhoff hypothesis. Although a functional relation is presumed to exist between the stress and strain components, none is explicitly used to develop the theory, in keeping with the general nature of these chapters.

Chapter 7 is more directly concerned with REPSIL. First the general theory developed previously is specialized to thin shells and the rotatory inertia is neglected. Moreover, the stress - strain relation for any material composing the shell is assumed to be isotropic and linearly elastic - perfectly plastic. These assumptions give us a convenient version of the equations forming the mathematical basis of REPSIL, which are then used to illustrate the computational algorithm employed by REPSIL to solve problems.*

As mentioned earlier, Chapter 8 gives a comparison of REPSIL, PETROS 1 and PETROS 2. First the theoretical and then the computational differences are discussed. These are then summarized in tabular form.

The reader principally interested in Chapters 7 and 8 is advised to read at the very minimum Chapter 2 in order to acquaint himself with the differential geometric approach used in REPSIL, as well as in PETROS 1 and PETROS 2, and if further time permits, he should skim Chapters 3 through 5 to familiarize himself with the significance of the more commonly encountered terms. Chapter 6 can be safely skipped.

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*Although the algorithm is illustrated assuming perfectly plastic material behavior, actually REPSIL has the capability of treating strain hardening, strain rate sensitive materials through the use of the mechanical sublayer model described in [2; Sect. 5.4.2].

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2. GEOMETRY OF A DEFORMING SHELL

In this chapter we present the mathematics needed to describe the
motion of a shell geometrically. This involves introducing concepts
and results from differential geometry. We first treat the motion of
the reference surface of the shell, after which the consequences of
assuming the Kirchhoff hypothesis for the deformation through the shell
thickness are obtained.

2.1 Motion of the Reference Surface

A shell can be described qualitatively as a three-dimensioned body
having one of its dimensions, the thickness, small in comparison to its
other dimensions. Shell theory takes advantage of this property in
order to simplify the study of shells by reducing an essentially three-
dimensional theory to one defined on a two-dimensional subspace, i.e. a
surface. This is accomplished by assuming that the deformation through
the thickness is of a known functional form. Once this is done, we need
only study the deformation of a given surface embedded in the shell,
called the *reference surface*[*],and the associated changes in the shell
variables defined on this surface.

Denoting the surface parameters by ξ^{α}[**], the time by t and the
position vector in Euclidean 3-space relative to some fixed origin by

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[*] It is customary to pick the reference surface to be the middle surface
of the shell; that is, the surface equidistant from the bounding faces
of the shell. However, whether a surface initially equidistant from
the bounding faces remains a middle surface throughout the history of
the deformation is a function of the form assumed for the deformation
through tne thickness. Hence, when no form has been explicitly assumed,
the term reference surface is to be preferred.

[**] Index notation is employed throughout this report, with Greek indices
being tacitly assumed to range over the integers 1, 2. Hence,
$\xi^{\alpha} \equiv \xi^{1}, \xi^{2}$.

r , the motion of the reference surface is specified by the reference surface deformation function:

$$r = r\ (\xi^{\alpha}, t)\ ,\tag{2.1}$$

with ξ^{α} ranging over some connected region in the plane and t over some interval of the real line. We demand that the mapping $\xi^{\alpha} \to r$ be one-to-one for any time t. We also require that the function $r\ (\xi^{\alpha}, t)$ be sufficiently differentiable so as to guarantee the continuity of all derivatives subsequently introduced. Moreover, we assume for simplicity that the ξ^{α} are *material coordinates*; that is, for fixed ξ^{α} the image r of the point ξ^{α} under the mapping (2.1) represents the successive positions of the same *material point* or *particle* on the reference surface.

2.2 Differential Geometry of the Reference Surface

In this section we are concerned with the spatial characterization of the reference surface at some fixed instant of time. We will need to use the language of the differential geometry of surfaces in Euclidean 3-space. Our treatment of this subject is not meant to be exhaustive.[*] Rather, we aim to present only selected results with a minimum of proof. In this section we shall not bother to indicate the dependence of variables on t explicitly.

By holding the parameter ξ^{1} constant and allowing ξ^{2} to vary, we generate a member of the family of ξ^{1} = *const. coordinate curves*. Conversely, by holding ξ^{2} constant and varying ξ^{1} we generate a member of the family of ξ^{2} = *const. coordinate curves*. We require that these two families of curves be independent in the sense that no member of one be anywhere tangent to any member of the other. Thus, these two families define a curvilinear coordinate system on the reference surface.

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[*] Extended treatments of the differential geometry of surfaces can be found in numerous text books on tensor analysis and differential geometry, of which the following are a representative collection: EISENHART [6], SOKOLNIKOFF [7] and THOMAS [8].

The partial derivative of the function $\underset{\sim}{r}\,(\xi^\alpha)$ with respect to ξ^1 holding ξ^2 constant

$$\underset{\sim}{a}_1\,(\xi^\alpha) \equiv \underset{\sim}{r}\,,_1\,(\xi^\alpha)^* \tag{2.2}$$

defines a vector tangent to a $\xi^2 = \text{const}$ coordinate curve and the other partial derivative holding ξ^1 constant

$$\underset{\sim}{a}_2\,(\xi^\alpha) \equiv \underset{\sim}{r}\,,_2\,(\xi^\alpha) \tag{2.3}$$

a vector tangent to a $\xi^1 = \text{const}$ coordinate curve. We require that the vectors $\underset{\sim}{a}_\alpha$ nowhere vanish. Hence, by virtue of the previously assumed independence of the families of coordinate curves, the pair $\underset{\sim}{a}_\alpha$ form a linearly independent set of vectors tangent to the reference surface and hence serve as a *basis* for vectors tangent to the reference surface. They are in fact the basis generated by the coordinate curves.

The unit *normal* to the surface is simply

$$\underset{\sim}{n}\,(\xi^\alpha) = \frac{\underset{\sim}{a}_1 \times \underset{\sim}{a}_2}{\left|\underset{\sim}{a}_1 \times \underset{\sim}{a}_2\right|} \;. \tag{2.4}$$

Clearly the triad $(\underset{\sim}{a}_\alpha, \underset{\sim}{n})$ forms a basis for vectors in 3-space, with the special orthogonality property that

$$\underset{\sim}{n} \cdot \underset{\sim}{a}_\alpha = 0 \;. \tag{2.5}$$

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*We use the comma notation for partial derivatives with respect to material coordinates: a partial derivative with respect to ξ^α is denoted by a subscripted comma followed by the index α ; that is,

$$\frac{\partial}{\partial \xi^\alpha} \equiv \,,_\alpha \;\;.$$

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The *covariant components of the metric or the first fundamental
tensor* of the reference surface are defined as

$$a_{\alpha\beta} = \underline{a}_\alpha \cdot \underline{a}_\beta \quad . \tag{2.6}$$

They form a symmetric and positive matrix. Denoting the determinant of
the metric matrix by a, we have

$$a \equiv \text{Det } a_{\alpha\beta} = a_{11}a_{22} - (a_{12})^2 = |\underline{a}_1 \times \underline{a}_2|^2 > 0 \quad . \tag{2.7}$$

The *contravariant components of the surface metric* can then be defined
as

$$a^{11} = a_{22}/a \ , \quad a^{12} = -a_{12}/a = a^{21} \ , \quad a^{22} = a_{11}/a \ , \tag{2.8}$$

and they also form a symmetric and positive matrix. Moreover, $a^{\alpha\beta}$ is
the *inverse* of $a_{\alpha\beta}$:

$$a_{\alpha\gamma}a^{\gamma\beta} \equiv a_{\alpha\gamma}a^{\beta\gamma} \equiv a_{\gamma\alpha}a^{\gamma\beta} = \delta_\alpha^{\beta*} \quad , \tag{2.9}$$

with δ_β^α (= 1 when $\alpha = \beta$, = 0 when $\alpha \neq \beta$) the Kronecker delta. By em-
ploying the contravariant components of the metric tensor, the *dual
basis* is generated:

$$\underline{a}^\alpha = a^{\alpha\beta}\underline{a}_\beta \quad . \tag{2.10}$$

The dual is inversely related to the basis $\underline{a}_\alpha$ and is also normal to the

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*We employ the summation convention: terms having a repeated index, once
as a subscript and once as a superscript, are to be summed over the
range of that index.

surface normal n:

$$a^\alpha \cdot a_\beta = \delta^\alpha_\beta \quad , \quad a^\alpha \cdot n = 0 . \tag{2.11}$$

The covariant and contravariant components of the *skew-symmetric surface tensor* ε^* are defined as

$$\varepsilon_{\alpha\beta} = a^{\frac{1}{2}} e_{\alpha\beta} \quad , \quad \varepsilon^{\alpha\beta} = a^{-\frac{1}{2}} e^{\alpha\beta} , \tag{2.12}$$

where $e_{\alpha\beta} \equiv e^{\alpha\beta}$ (= 1 when $\alpha = 1$ & $\beta = 2$, = -1 when $\alpha = 2$ & $\beta = 1$, = 0 when $\alpha = \beta$) is the permutation symbol. The components of ε and the Kronecker delta satisfy the usual relations:

$$\varepsilon^{\alpha\beta}\varepsilon_{\gamma\delta} = \delta^\alpha_\gamma \delta^\beta_\delta - \delta^\alpha_\delta \delta^\beta_\gamma \quad , \quad \varepsilon^{\alpha\gamma}\varepsilon_{\beta\gamma} = \delta^\alpha_\beta \quad , \quad \varepsilon^{\gamma\delta}\varepsilon_{\gamma\delta} = 2 . \tag{2.13}$$

Using the ε tensor, we can summarize the relationships between the basis a_α, the dual basis a^α and the surface normal n in the following compact forms:

$$a_\alpha \times a_\beta = \varepsilon_{\alpha\beta}n \quad , \quad a^\alpha \times a^\beta = \varepsilon^{\alpha\beta}n \quad , \tag{2.14}$$

$$n \times a_\alpha = \varepsilon_{\alpha\beta}a^\beta \quad , \quad n \times a^\alpha = \varepsilon^{\alpha\beta}a_\beta .$$

A tensor as important as the metric or first fundamental tensor for surfaces embedded in 3-space is the *second fundamental tensor*, which measures the spatial rate of change of the normal n with respect to the surface basis. Its covariant components are defined as

$$b_{\alpha\beta} = - a_\alpha \cdot n,_\beta = n \cdot a_{\alpha},_\beta = n \cdot r,_{\alpha\beta} \quad , \tag{2.15}$$

ε^* is only a tensor under coordinate transformations with positive Jacobians since the permutation symbol $e_{\alpha\beta}$ transforms like a relative tensor of weight ±1; cf. SOKOLNIKOFF [7; p. 107].

the last equality clearly showing that the $\underline{b}$ tensor is symmetric. Similarly, the components of the *connection* for the surface are defined as

$$\Gamma_{\alpha\beta}{}^{\gamma} = \underline{a}^{\gamma} \cdot \underline{a}_{\alpha,\beta} = \underline{a}^{\gamma} \cdot \underline{r}_{,\alpha\beta} \quad , \tag{2.16}$$

with the symmetry with respect to the first and second indices exhibited by the last equality. A standard reduction[*] relates the components of the connection to the partials of the metric components

$$\Gamma_{\alpha\beta}{}^{\gamma} = \frac{1}{2} a^{\gamma\delta} (a_{\beta\delta,\alpha} + a_{\delta\alpha,\beta} - a_{\alpha\beta,\delta}) \; . \tag{2.17}$$

Hence, the components of the connection are the Christoffel symbols. It easily follows from this that

$$a^{\frac{1}{2}}{}_{,\alpha} = \frac{1}{2} a^{\frac{1}{2}} a^{\beta\gamma} a_{\beta\gamma,\alpha} = a^{\frac{1}{2}} \Gamma_{\alpha\beta}{}^{\beta} \; . \tag{2.18}$$

The covariant derivative of any quantity with respect to ξ^{α}, denoted by an attached subscript $|\alpha$, is defined in the usual way through the connection.[**] The surface metric and the surface tensor $\underline{\varepsilon}$ have the property that the covariant derivative of their components identically vanish:

$$a_{\alpha\beta}{}_{|\gamma} = 0 \quad , \quad a^{\alpha\beta}{}_{|\gamma} = 0 \quad , \quad \varepsilon_{\alpha\beta}{}_{|\gamma} = 0 \quad , \quad \varepsilon^{\alpha\beta}{}_{|\gamma} = 0. \tag{2.19}$$

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[*] See SOKOLNIKOFF [7; p. 129].

[**] For example, the covariant derivative of the two index quantity $\delta^{\beta}{}_{\gamma}$ with respect to ξ^{α} is

$$\delta^{\beta}{}_{\gamma|\alpha} \equiv \delta^{\beta}{}_{\gamma,\alpha} + \delta^{\delta}{}_{\gamma} \Gamma_{\alpha\delta}{}^{\beta} - \delta^{\beta}{}_{\delta} \Gamma_{\alpha\gamma}{}^{\delta} \quad ,$$

cf. EISENHART [6; p. 110] or THOMAS [8; p.54].

This well known property shall be freely used throughout. It easily follows from this property and the definition of the second fundamental tensor that

$$\underset{\sim}{a}_{\alpha \mid \beta} = b_{\alpha\beta} \, \underset{\sim}{n} \quad , \quad \underset{\sim}{a}^{\alpha}{}_{\mid \beta} = b^{\alpha}_{\beta} \, \underset{\sim}{n} \quad , \quad \underset{\sim}{n}_{\mid \beta} = - b^{\alpha}_{\beta} \, \underset{\sim}{a}_{\alpha} \quad . \qquad (2.20)$$

Note that the components of the metric are used to raise and lower the indices of the second fundamental tensor (i.e. $b^{\alpha}_{\beta} = a^{\alpha\gamma} b_{\gamma\beta}$), as is commonly done with the components of any tensor associated with the surface. Lastly, we make mention of the fact that due to the second fundamental tensor and the connection being defined as derivatives of the basis $\underset{\sim}{a}_{\alpha}$ they satisfy the Codazzi and Gauss equations[*]

$$b_{\alpha\beta \mid \gamma} - b_{\alpha\gamma \mid \beta} = 0 \quad , \quad R_{\alpha\beta\gamma\delta} = b_{\alpha\delta} b_{\beta\gamma} - b_{\alpha\gamma} b_{\beta\delta} \quad , \qquad (2.21)$$

where

$$R_{\alpha\beta\gamma\delta} = a_{\alpha\nu} \, (\Gamma_{\gamma\beta}{}^{\nu}{}_{,\delta} - \Gamma_{\delta\beta}{}^{\nu}{}_{,\gamma} + \Gamma_{\gamma\beta}{}^{\mu} \Gamma_{\delta\mu}{}^{\nu} - \Gamma_{\delta\beta}{}^{\mu} \Gamma_{\gamma\mu}{}^{\nu}) \qquad (2.22)$$

is the Riemann Christoffel curvature tensor for the surface.

2.3 Kinematics of the Reference Surface

The time derivative holding the material coordinates constant is the *material derivative*. The material derivative of the function $\underset{\sim}{r} \, (\xi^{\alpha}, t)$ is the *velocity* at the time t of the particle with material coordinates ξ^{α}:

$$\underset{\sim}{v} \, (\xi^{\alpha}, t) = \dot{\underset{\sim}{r}} \, (\xi^{\alpha}, t) \, . \qquad \qquad (2.23)$$

[**]

Resolved into normal and tangential components, the velocity becomes

$$\underset{\sim}{v} = v^{\alpha} \, \underset{\sim}{a}_{\alpha} + v \, \underset{\sim}{n} \quad . \qquad (2.24)$$

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[*] Cf. EISENHART [6; p. 219] or THOMAS [8; p. 89].

[**] A superposed dot denotes the material derivative.

The covariant derivative of the velocity, the *velocity gradient*, is also resolved into normal and tangential components

$$\underset{\sim}{v}_{|\alpha} \equiv \underset{\sim}{v}_{,\alpha} = (v^\beta{}_{|\alpha} - b^\beta_\alpha v) \underset{\sim}{a}_\beta + (v_{|\alpha} + b_{\alpha\beta} v^\beta) \underset{\sim}{n} , \qquad (2.25)$$

using relations (2.20). A reversal of order of differentiation shows that the velocity gradient is the material derivative of the basis:

$$\underset{\sim}{v}_{|\alpha} = \underset{\sim}{\mathring{a}}_\alpha . \qquad (2.26)$$

The divergence of the velocity $\underset{\sim}{v}$ can be written as

$$\nabla \cdot \underset{\sim}{v} \equiv \underset{\sim}{a}^\alpha \cdot \underset{\sim}{v}_{|\alpha} = v^\alpha{}_{|\alpha} - b^\alpha_\alpha v . \qquad (2.27)$$

Combining the last two equations, we obtain the useful relation

$$\mathring{a}^{\frac{1}{2}} = \frac{1}{2} a^{\frac{1}{2}} a^{\alpha\beta} \mathring{a}_{\alpha\beta} = a^{\frac{1}{2}} \nabla \cdot \underset{\sim}{v} . \qquad (2.28)$$

Since the normal vector $\underset{\sim}{n}$ is of constant (unit) magnitude, its material derivative $\underset{\sim}{\omega}$ will be tangential to the surface and hence have the resolution:

$$\underset{\sim}{\mathring{n}} = \underset{\sim}{\omega} = \omega^\alpha \underset{\sim}{a}_\alpha = \omega_\alpha \underset{\sim}{a}^\alpha . \qquad (2.29)$$

Also, since $\underset{\sim}{n}$ and $\underset{\sim}{a}_\alpha$ are normal, their material derivatives cannot be totally independent, but must, indeed, satisfy the relation

$$\underset{\sim}{\omega} \cdot \underset{\sim}{a}_\alpha + \underset{\sim}{n} \cdot \underset{\sim}{v}_{|\alpha} = 0 , \qquad (2.30)$$

from which it follows that

$$\underset{\sim}{\omega} = - \underset{\sim}{n} \cdot \underset{\sim}{v}_{|\alpha} \underset{\sim}{a}^\alpha , \qquad (2.31)$$

or, using (2.25) and (2.29),

$$\omega_\alpha = - (v_{|\alpha} + b_{\alpha\beta} v^\beta) . \qquad (2.32)$$

The latter allows us to write the velocity gradient in the form

$$v_{|\alpha} = (v^\beta_{|\alpha} - b^\beta_\alpha v) a_\beta - \omega_\alpha n \quad . \tag{2.33}$$

Interesting results could now be obtained relating the material derivatives of the metric, second fundamental tensor, connection, etc. to various covariant derivatives of the velocity components, but since the material presented suffices for subsequent sections, we will not pursue this line.

2.4 Kirchhoff Hypothesis

The *Kirchhoff hypothesis* assumes that the shell deforms in such a way that particles initially on a normal to the reference surface will find themselves after the shell is deformed on the same normal with their distance from the reference surface unchanged. This hypothesis is an idealization of the observation that in many situations very little shearing or extension occurs through the thickness over major portions of the shell during deformation.

We shall assume that the shell is of uniform thickness and identify the reference surface with the middle surface of the shell, although any surface parallel to the middle surface would give an equivalent theory. Using the coordinate system introduced previously, we have by the Kirchhoff hypothesis that a particle located a distance ζ from the middle surface along the normal n passing through the middle surface particle with coordinates ξ^α, will always be situated on the normal through the same middle surface particle at the same distance ζ. Hence for all time the position of a particle is specified by the equation.

$$x = r (\xi^c,t) + \zeta n (\xi^\alpha,t) \quad . \tag{2.34}$$

as indicated in Figure 2.1.

Notice that a particle off the middle surface will always have associated with it the same coordinates (ξ^α,ζ), with $\zeta \neq 0$. Moreover, a particle on the middle surface is always associated with the coordinates

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Undeformed Position

Deformed Position
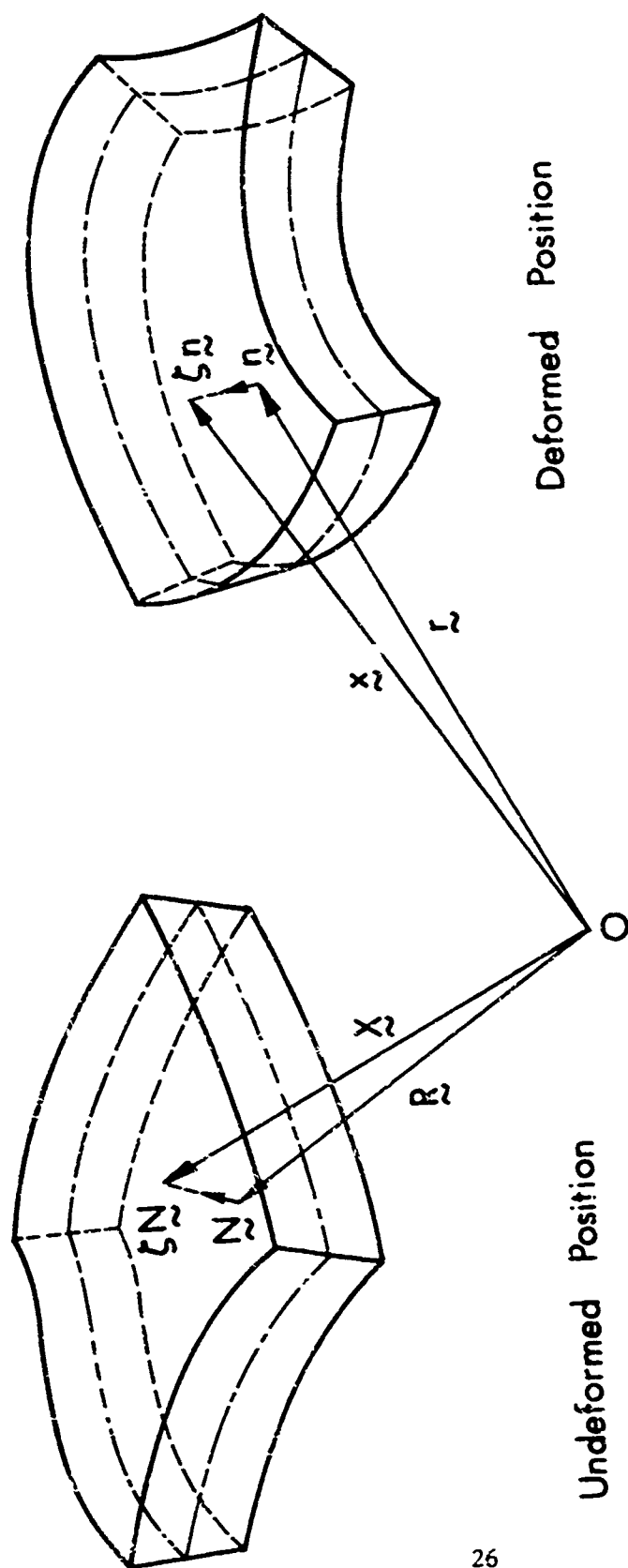
Figure 2.1 The deformation of a shell under the Kirchhoff hypothesis. A displacement from $\underset{\sim}{R}$ to $\underset{\sim}{r}$ of a particle on the middle surface gives rise to a simultaneous displace from $\underset{\sim}{X}$ to $\underset{\sim}{x}$ of the particle located a distance ζ along the normal from the middle surface particle.

$(\xi^\alpha, 0)$. Hence the triplex (ξ^α, ζ) defines a material coordinate system for all the particles composing the shell. Taking h to be th uniform thickness dimension of the shell, ζ will range between the limits $\pm \frac{h}{2}$, while as before ξ^α range over some suitable connected region in the plane.

Definitions (2.23) and (2.29) allow us to write the particle velocity as

$$\dot{x} = v + \zeta \, \omega \quad . \tag{2.35}$$

The derivatives of (2.34) with respect to material coordinates (ξ^α, ζ) define a basis in 3-space:

$$g_\beta = x,_\beta \quad , \quad g_3 = \frac{\partial x}{\partial \zeta} \quad . \tag{2.36}$$

From $(2.20)_3$ it follows that the basis g_i * is related to the basis (a_α, n) through the relations

$$g_\beta = \mu_\beta^\gamma \, a_\gamma \quad , \quad g_3 = n \quad , \tag{2.37}$$

where the convenient abbreviation

$$\mu_\beta^\gamma = \delta_\beta^\gamma - \zeta \, b_\beta^\gamma \tag{2.38}$$

is employed. The pair $g_\beta (\xi^\beta, \zeta)$ defines a basis for the lamella parallel to the middle surface at a distance ζ. When $\zeta = 0$ the basis g_β reduces to the middle surface basis a_β. Notice that n is normal to each lamella, since

$$n \cdot g_\alpha = 0 \quad . \tag{2.39}$$

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* Latin indices are introduced for the range 1, 2, 3.

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A metric for that portion of space occupied by the shell can be specified using the basis $\underset{\sim}{g_i}$ as follows

$$g_{ij} = \underset{\sim}{g_i} \cdot \underset{\sim}{g_j} \quad . \tag{2.40}$$

The determinant and inverse of the metric are then defined in the usual way

$$g = \text{Det } g_{ij} \equiv \frac{1}{6} e^{ijk} e^{lmn} g_{il} g_{jm} g_{kn} \quad ,$$

$$g^{ij} = \frac{1}{2g} e^{ike} e^{jmn} g_{km} g_{ln} \quad , \tag{2.41}$$

where e^{ijk} (= 1 when i, j, k is an even permutation of 1, 2, 3; = -1 when i, j, k is an odd permutation; = 0 when i, j, k not distinct) is the 3-dimensional permutation symbol. The inverse allows us to define the dual basis

$$\underset{\sim}{g^i} = g^{ij} \underset{\sim}{g_j} \quad , \tag{2.42}$$

having the reciprocal property

$$\underset{\sim}{g^i} \cdot \underset{\sim}{g_j} = \delta^i_j \quad . \tag{2.43}$$

The metric and its inverse assume the simple forms

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad g^{ij} = \begin{pmatrix} g^{11} & g^{12} & 0 \\ g^{21} & g^{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.44}$$

due to $\underset{\sim}{g_3}$ being a unit vector normal to $\underset{\sim}{g_\alpha}$. Hence we need only work with the metric $g_{\alpha\beta}$ of lamellar parallel to the middle surface and its inverse $g^{\alpha\beta}$. Moreover the determinant g and dual basis $\underset{\sim}{g^i}$ become

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$$g = \text{Det } g_{\alpha\beta} = \frac{1}{2} e^{\alpha\beta} e^{\gamma\delta} g_{\alpha\gamma} g_{\beta\delta} \quad ,$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.45)$$

$$g^{\alpha} = g^{\alpha\beta} g_{\beta} \quad , \qquad g^{3} = g_{3} = n \quad .$$

Using (2.37), we relate the above quantities to the corresponding middle surface quantities

$$g_{\alpha\beta} = \mu_{\alpha}^{\gamma} \mu_{\beta}^{\delta} a_{\gamma\delta} \quad , \qquad g^{\alpha\beta} = \overset{-1\alpha}{\mu}_{\beta} \overset{-1\beta}{\mu}_{\delta} a^{\gamma\delta} \quad ,$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.46)$$

$$g = (\mu)^{2} a \qquad , \qquad g^{\alpha} = \overset{-1\alpha}{\mu}_{\beta} a^{\beta} \qquad ,$$

where μ and $\overset{-1\alpha}{\mu}_{\beta}$ denote the determinant and inverse of μ_{β}^{α} :

$$\mu = \frac{1}{2} e^{\alpha\gamma} e_{\beta\delta} \mu_{\alpha}^{\beta} \mu_{\gamma}^{\delta} \quad , \quad \overset{-1\alpha}{\mu}_{\beta} = \frac{1}{\mu} e^{\alpha\gamma} e_{\beta\delta} \mu_{\nu}^{\delta} \quad . \qquad (2.47)$$

Clearly, the basis g_{α} , its dual g^{α} , the metric $g_{\alpha\beta}$ and its inverse only exist when $\mu \neq 0$. It can be shown* that a sufficient condition to guarantee existence for ζ in the range $\pm \frac{h}{2}$ is $h < 2 R_{min}$, where R_{min} is the minimum principle radius of curvature of the middle surface. With this condition $\mu > 0$. We shall assume this mild condition to hold in developing the theory of general Kirchhoff shells, in Chapters 2 through 6. In specializing the general theory to that forming the basis for REPSIL in Chapter 7, we go further and require that the more stringent thin shell hypothesis holds, wherein $h \ll R_{min}$, so that terms of order $|\zeta b_{\beta}^{\alpha}|^{2}$ can be neglected in comparison with unity.

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* See NAGHDI [9; p 17].

3. STRAIN IN A SHELL

In this chapter a measure of strain is defined based on the change
in the metric. This measure is simplified using the Kirchhoff hypothesis.
Equations are next obtained relating the incremental change in strain
to the gradients of the displacement increments of the middle surface.

3.1 Definition of Strain

Strain is any quantity measuring the amount of stretching that
occurs in the local neighborhood of a particle during deformation. The
differential vector connecting the position $r (\xi^{\alpha}, \zeta)$ of a particle with
the position $r (\xi^{\alpha} + d\xi^{\alpha}, \zeta + d\zeta)$ of any neighboring particle is given
by

$$dr = g_{\alpha} \, d\xi^{\alpha} + g_{3} \, d\zeta \quad . \tag{3.1}$$

The distance ds between these particles is the magnitude of dr , or

$$ds^2 = g_{\alpha\beta} \, d\xi^{\alpha} \, d\xi^{\beta} + (g_{\alpha 3} + g_{3\alpha}) \, d\xi^{\alpha} \, d\zeta + g_{33} \, d\zeta^2 \quad . \tag{3.2}$$

Initially at time $t = t_{o}$ this distance is given as

$$dS^2 = G_{\alpha\beta} \, d\xi^{\alpha} \, d\xi^{\beta} + (G_{\alpha 3} + G_{3\alpha}) \, d\xi^{\alpha} \, d\zeta + G_{33} \, d\zeta^2 \quad .^{*} \tag{3.3}$$

A measure of the stretching undergone from the time $t = t_{o}$ is

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*Throughout this report the initial values of variables when $t = t_{o}$
will be denoted by replacing minuscule by corresponding majuscule
letters, whenever this can be done without ambiguity. Otherwise a
zero subscript will be appended to indicate initial values.

$$\frac{1}{2} (ds^2 - dS^2) = E_{\alpha\beta} \, d\xi^\alpha \, d\xi^\beta + (E_{\alpha 3} + E_{3\alpha}) \, d\xi^\alpha \, d\zeta + E_{33} \, d\zeta^2 \, , \quad (3.4)$$

where

$$E_{ij} = \frac{1}{2} (g_{ij} - G_{ij}) \, . \qquad\qquad (3.5)$$

This measure vanishes when no stretching occurs between two particles. Moreover when no stretching occurs in the neighborhood of a particle, so that the measure of stretching vanishes for all $d\xi^\alpha$, $d\zeta$, then E_{ij} will vanish. This makes it reasonable to adopt E_{ij} as a strain measure.

Due to the Kirchhoff hypothesis, $g_{\alpha 3} = g_{3\alpha} = 0$ and $g_{33} = 1$ for all time. Consequently, the strain E_{ij} will assume the simple form

$$E_{ij} = \begin{pmatrix} E_{11} & E_{12} & 0 \\ E_{21} & E_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \, . \qquad\qquad (3.6)$$

This shows that no stretching or shearing takes place in the transverse direction. Hence we need only consider the nonzero components of the strain, namely

$$E_{\alpha\beta} = \frac{1}{2} (g_{\alpha\beta} - G_{\alpha\beta}) \, . \qquad\qquad (3.7)$$

Clearly, the symmetry of $g_{\alpha\beta}$ implies the symmetry of $E_{\alpha\beta}$.

By virtue of (2.58) and (2.46)$_1$ the nonzero components of strain can be related to the surface metric and b tensor:

$$E_{\alpha\beta} = \frac{1}{2} [(a_{\alpha\beta} - A_{\alpha\beta}) - 2\zeta (b_{\alpha\beta} - B_{\alpha\beta}) + \zeta^2 (b^2_{\alpha\beta} - B^2_{\alpha\beta})] \, , \quad (3.8)$$

where $b^2_{\alpha\beta}$ denotes the components of the square of the second fundamental tensor:

$$b^2_{\alpha\beta} = a_{\gamma\delta} \, b^\gamma_\alpha \, b^\delta_\beta \, . \qquad\qquad (3.9)$$

From the definition of $b_{\alpha\beta}$ (2.15) and the fact that $\underset{\sim}{n}$ is a unit vector, it follows that $b^2_{\alpha\beta}$ can be written as

$$b^2_{\alpha\beta} = \underset{\sim}{n}{,}_\alpha \cdot \underset{\sim}{n}{,}_\beta = - \underset{\sim}{n} \cdot \underset{\sim}{n}{,}_{\alpha\beta} \tag{3.10}$$

Clearly $b^2_{\alpha\beta}$ is symmetric.

3.2 Strain - Displacement Relations

Let $\Delta\underset{\sim}{u}$ be the displacement undergone by a middle surface particle from some past time t^- to the current time t. Denote the values of variables at the time t^- by an attached minus superscript and leave unchanged the values corresponding to t. Using this notation, the displacement $\Delta\underset{\sim}{u}$ becomes

$$\Delta\underset{\sim}{u} = \underset{\sim}{r} - \underset{\sim}{r}^- \quad . \tag{3.11}$$

The strains at these times are

$$E^-_{\alpha\beta} = \frac{1}{2} \left[(a^-_{\alpha\beta} - A_{\alpha\beta}) - 2\zeta (b^-_{\alpha\beta} - B_{\alpha\beta}) + \zeta^2 (b^{2-}_{\alpha\beta} - B^2_{\alpha\beta}) \right] ,$$

$$\tag{3.12}$$

$$E_{\alpha\beta} = \frac{1}{2} \left[(a_{\alpha\beta} - A_{\alpha\beta}) - 2\zeta (b_{\alpha\beta} - B_{\alpha\beta}) + \zeta^2 (b^2_{\alpha\beta} - B^2_{\alpha\beta}) \right] .$$

Hence, it follows that the finite strain increment is additive:

$$E_{\alpha\beta} = E^-_{\alpha\beta} + \Delta E_{\alpha\beta} \quad , \tag{3.13}$$

where the incremental strain is given by

$$\Delta E_{\alpha\beta} = \frac{1}{2} \left[\Delta a_{\alpha\beta} - 2\zeta \, \Delta b_{\alpha\beta} + \zeta^2 \, \Delta b^2_{\alpha\beta} \right] \tag{3.14}$$

and

$$\Delta a_{\alpha\beta} = a_{\alpha\beta} - a^-_{\alpha\beta}, \quad \Delta b_{\alpha\beta} = b_{\alpha\beta} - b^-_{\alpha\beta}, \quad \Delta b^2_{\alpha\beta} = b^2_{\alpha\beta} - b^{2-}_{\alpha\beta} . \tag{3.15}$$

In order to express the strain increment in terms of gradients of $\Delta \underset{\sim}{u}$ and $\Delta \underset{\sim}{n}$, we first observe that

$$\Delta \underset{\sim}{u}_{,\alpha} = \underset{\sim}{a}_{\alpha} - \bar{\underset{\sim}{a}}_{\alpha} \quad , \qquad \Delta \underset{\sim}{n} = \underset{\sim}{n} - \bar{\underset{\sim}{n}} \quad . \tag{3.16}$$

Hence, from the definitions of $a_{\alpha\beta}$, $b_{\alpha\beta}$ and $b^2_{\alpha\beta}$, equations (2.6), (2.15) and (3.10), we have

$$\Delta a_{\alpha\beta} = \bar{\underset{\sim}{a}}_{\alpha} \cdot \Delta \underset{\sim}{u}_{,\beta} + \bar{\underset{\sim}{a}}_{\beta} \cdot \Delta \underset{\sim}{u}_{,\alpha} + \Delta \underset{\sim}{u}_{,\alpha} \cdot \Delta \underset{\sim}{u}_{,\beta} \quad ,$$

$$\Delta b_{\alpha\beta} = \bar{\underset{\sim}{n}} \cdot \Delta \underset{\sim}{u}_{,\alpha\beta} + \Delta \underset{\sim}{n} \cdot \bar{\underset{\sim}{a}}_{\alpha,\beta} + \Delta \underset{\sim}{n} \cdot \Delta \underset{\sim}{u}_{,\alpha\beta} \quad , \tag{3.17}$$

$$\Delta b^2_{\alpha\beta} = \bar{\underset{\sim}{n}}_{,\alpha} \cdot \Delta \underset{\sim}{n}_{,\beta} + \bar{\underset{\sim}{n}}_{,\beta} \cdot \Delta \underset{\sim}{n}_{,\alpha} + \Delta \underset{\sim}{n}_{,\alpha} \cdot \Delta \underset{\sim}{n}_{,\beta} \quad ,$$

or combining differently, we also have

$$\Delta a_{\alpha\beta} = \underset{\sim}{a}_{\alpha} \cdot \Delta \underset{\sim}{u}_{,\beta} + \underset{\sim}{a}_{\beta} \cdot \Delta \underset{\sim}{u}_{,\alpha} - \Delta \underset{\sim}{u}_{,\alpha} \cdot \Delta \underset{\sim}{u}_{,\beta} \quad ,$$

$$\Delta b_{\alpha\beta} = \underset{\sim}{n} \cdot \Delta \underset{\sim}{u}_{,\alpha\beta} + \Delta \underset{\sim}{n} \cdot \underset{\sim}{a}_{\alpha,\beta} - \Delta \underset{\sim}{n} \cdot \Delta \underset{\sim}{u}_{,\alpha\beta} \quad , \tag{3.18}$$

$$\Delta b^2_{\alpha\beta} = \underset{\sim}{n}_{,\alpha} \cdot \Delta \underset{\sim}{n}_{,\beta} + \underset{\sim}{n}_{,\beta} \cdot \Delta \underset{\sim}{n}_{,\alpha} - \Delta \underset{\sim}{n}_{,\alpha} \cdot \Delta \underset{\sim}{n}_{,\beta} \quad .$$

With these two sets of expressions we can write the strain increment in the following equivalent ways:

$$\Delta E_{\alpha\beta} = \frac{1}{2} \left[(\bar{\underset{\sim}{a}}_{\alpha} \cdot \Delta \underset{\sim}{u}_{,\beta} + \bar{\underset{\sim}{a}}_{\beta} \cdot \Delta \underset{\sim}{u}_{,\alpha} + \Delta \underset{\sim}{u}_{,\alpha} \cdot \Delta \underset{\sim}{u}_{,\beta}) \right.$$

$$-2\zeta \, (\bar{\underset{\sim}{n}} \cdot \Delta \underset{\sim}{u}_{,\alpha\beta} + \Delta \underset{\sim}{n} \cdot \bar{\underset{\sim}{a}}_{\alpha,\beta} + \Delta \underset{\sim}{n} \cdot \Delta \underset{\sim}{u}_{,\alpha\beta}) \tag{3.19}$$

$$\left. +\zeta^2 \, (\bar{\underset{\sim}{n}}_{,\alpha} \cdot \Delta \underset{\sim}{n}_{,\beta} + \bar{\underset{\sim}{n}}_{,\beta} \cdot \Delta \underset{\sim}{n}_{,\alpha} + \underset{\sim}{n}_{,\alpha} \cdot \Delta \underset{\sim}{n}_{,\beta}) \right]$$

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and

$$\Delta E_{\alpha\beta} = \frac{1}{2} \left[(a_{\alpha} \cdot \Delta u_{,\beta} + a_{\beta} \cdot \Delta u_{,\alpha} - \Delta u_{,\alpha} \cdot \Delta u_{,\beta}) \right.$$

$$-2\zeta \, (n \cdot \Delta u_{,\alpha\beta} + \Delta n \cdot a_{\alpha,\beta} - \Delta n \cdot \Delta u_{,\alpha\beta}) \qquad (3.20)$$

$$\left. +\zeta^2 \, (n_{,\alpha} \cdot \Delta n_{,\beta} + n_{,\beta} \cdot \Delta n_{,\alpha} - \Delta n_{,\alpha} \cdot \Delta n_{,\beta}) \right] \quad .$$

Both these expressions are exact within the limits of the Kirchhoff hypothesis. They are quadratic in increments, with no linearization based on the smallness of these increments being used. An exact expression linear in the increments can be obtained by averaging these expressions:

$$\Delta E_{\alpha\beta} = \frac{1}{2} \left[(\bar{a}_{\alpha} \cdot \Delta u_{,\beta} + \bar{a}_{\beta} \cdot \Delta u_{,\alpha}) \right.$$

$$-2\zeta \, (\bar{n} \cdot \Delta u_{,\alpha\beta} + \Delta n \cdot \bar{a}_{\alpha,\beta}) \qquad (3.21)$$

$$\left. +\zeta^2 \, (\bar{n}_{,\alpha} \cdot \Delta n_{,\beta} + \bar{n}_{,\beta} \cdot \Delta n_{,\alpha}) \right] \quad ,$$

with

$$\bar{a}_{\alpha} = \frac{1}{2} (a_{\alpha} + a_{\alpha}^{-}) \quad , \qquad \bar{n} = \frac{1}{2} (n + n^{-}) \quad . \qquad (3.22)$$

It should be noted that the last quantities have no significance other than as averages, there not necessarily being any intermediate position of the middle surface for which $\bar{a}_{\alpha}$ forms a basis and $\bar{n}$ a surface normal. Indeed, $\bar{n}$ will not generally be a unit vector nor be normal to $\bar{a}_{\alpha}$.

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3.3 Surface Normal - Displacement Relations

When the gradient of the displacement increment $\Delta u_{\sim,\alpha}$ is small, there is very little difference between n^- and n. This makes the use of $(3.16)_2$ unsatisfactory for determining Δn numerically, since computational accuracy could be impaired, and suggests the use of a linear expression based on expanding Δn as a series in $\Delta u_{\sim,\alpha}$. On the other hand, for moderately large $\Delta u_{\sim,\alpha}$ the linear expression would lose accuracy. Since our aim is to ultimately use this theory in a computer code, these considerations lead us to derive an exact expression for Δn containing $\Delta u_{\sim,\alpha}$ as a linear factor.[*]

We begin by noting that due to n being a unit vector

$$\Delta n \cdot (n + n^-) \equiv (n - n^-) \cdot (n + n^-) = 0 \quad , \qquad (3.23)$$

and due to n being normal to a_α

$$\Delta n \cdot a_\alpha = - n^- \cdot a_\alpha = - n^- \cdot \Delta u_{\sim,\alpha} \quad . \qquad (3.24)$$

Resolving Δn relative to the basis (a^α, n), we obtain

$$\Delta n = \Delta n_\alpha a^\alpha + \Delta n\, n \quad . \qquad (3.25)$$

The tangential component,

$$\Delta n_\alpha = \Delta n \cdot a_\alpha \quad , \qquad (3.26)$$

can be written as a linear expression in $\Delta u_{\sim,\alpha}$ by using (3.24)

$$\Delta n_\alpha = - n^- \cdot \Delta u_{\sim,\alpha} \quad . \qquad (3.27)$$

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[*] As far as the author is aware, MORINO, LEECH and WITMER [2] are the first to have addressed themselves to this problem. The present analysis is partially based on theirs.

In order to cbtain an appropriate expression for the normal componet of Δn, we multiply (3.25) by $(\underline{n} + \underline{n}^-)$. By virtue of (3.23), the product vanishes so that

$$\Delta n_\alpha \, \underline{n}^- \cdot \underline{a}^\alpha + (1 + \underline{n} \cdot \underline{n}^-) \, \Delta n = 0 \quad . \tag{3.28}$$

Solving this equation for Δn and using (3.24) and (3.26), we obtain an expression quadratic in Δn_α and hence quadratic in $\Delta \underline{u}_{,\alpha}$:

$$\Delta n = \frac{a^{\alpha\beta} \, \Delta n_\alpha \, \Delta n_\beta}{1 + \underline{n} \cdot \underline{n}^-} \quad . \tag{3.29}$$

To insure that the denominator does not get too small and thus reduce computational accuracy, we require that the above expression for Δn be used only when the mild restriction $\underline{n} \cdot \underline{n}^- \geq 0$ is satisfied, otherwise $(3.16)_2$ should be used.

Rearranging slightly, we can summarize as follows: when $\underline{n} \cdot \underline{n}^- \geq 0$, determine the increment $\Delta \underline{n}$ using

$$\Delta \underline{n} = [\underline{a}_\alpha + \frac{\Delta n_\alpha}{1 + \underline{n} \cdot \underline{n}^-} \, \underline{n}] \, \Delta n^\alpha \quad , \tag{3.30}$$

with

$$\Delta n_\alpha = - \underline{n}^- \cdot \Delta \underline{u}_{,\alpha} \quad , \quad \Delta n^\alpha = a^{\alpha\beta} \, \Delta n_\beta \quad ; \tag{3.31}$$

otherwise when $\underline{n} \cdot \underline{n}^- < 0$, use $(3.16)_2$. Lastly, had the basis $(\underline{a}_\alpha^-, \, \underline{n}^-)$ been used to resolve $\Delta \underline{n}$, the expression for $\Delta \underline{n}$ would have assumed the similar form:

$$\Delta \underline{n} = [\underline{a}_\alpha^- + \frac{\Delta n_\alpha}{1 + \underline{n} \cdot \underline{n}^-} \, \underline{n}^-] \, \Delta n^\alpha \quad , \tag{3.32}$$

where now

$$\Delta n_\alpha = - \underline{n} \cdot \Delta \underline{u}_{,\alpha} \quad , \quad \Delta n^\alpha = a^{-\alpha\beta} \, \Delta n_\beta \quad . \tag{3.33}$$

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4. RESULTANT RELATIONS FOR MIDDLE SURFACE VARIABLES

Under the assumption that the shell deforms according to the Kirchhoff hypothesis, we derive in this chapter relations connecting the variables defined on the middle surface to associated variables defined on the shell. More specifically, we obtain the relations connecting $1°$ the linear momentum and spin momentum densities acting on the middle surface to the particle velocity field, $2°$ the stress and couple resultants to the stress field, and $3°$ the body force and body couple resultants to the body force field and tractions on the bounding surfaces. For the derivation we also assume that the material composing the shell is non-polar[*] and that the middle surface variables are resultants of the corresponding quantities defined on the shell.

4.1 Integration Through the Shell Thickness

In order to obtain the resultant on the middle surface of a variable defined on the shell, we need to express the volume integral as an integral over the middle surface of an integral through the thickness. We begin by letting S be any region in the middle surface bounded by the simple closed curve C. Let V be the portion of the shell generated by varying ζ between the limits $\pm\frac{h}{2}$ as ξ^α ranges over S, see Figure 4.1. The volume V is bounded by two surfaces S_+ (where $\zeta = \frac{h}{2}$), and S_- (where $\zeta = -\frac{h}{2}$), and by the surface S_c generated by the bounding curve C as ζ ranges between its limits. Using the basis g_{-i}, we can write the volume integral of any variable $\oint (\xi^\alpha, \zeta)$ defined on V as

$$\iiint\limits_V \oint (\xi^\alpha, \zeta)\ dv = \iint\limits_\Omega \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \oint (\xi^\alpha, \zeta)\ g^{\frac{1}{2}}\ d\zeta\right]\ d\xi^1\ d\xi^{2^{**}} , \qquad (4.1)$$

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[*] See NOLL & TRUESDELL [10; Sec 98].

[**] See SOKOLNIKOFF [7; Eq 44.11].
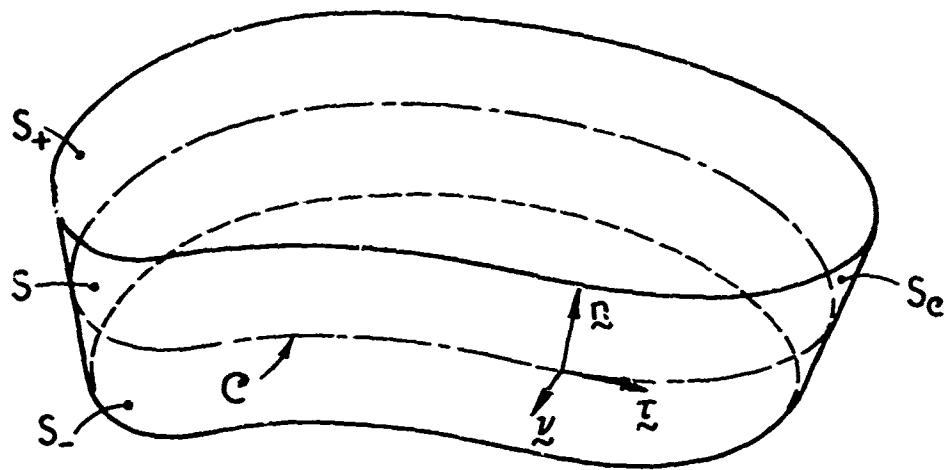
Figure 4.1 The volume V generated by varying ζ between the limits $\pm \frac{h}{2}$ over the region S of the middle surface. S is bounded by the curve C and V by the surfaces S_c, S_+ and S_-.

wherein Ω is the pre-image of S (see Appendix A). By virtue of $(2.46)_3$ the last expression becomes

$$\iiint\limits_V \oint (\xi^\alpha, \zeta) \, dv = \iint\limits_\Omega \left[\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \oint (\xi^\alpha, \zeta) \, \mu \, d\zeta \right] a^{\frac{1}{2}} \, d\zeta^1 \, d\zeta^2 \qquad (4.2)$$

or, since S is the image of Ω ,

$$\iiint\limits_V \oint (\xi^\alpha, \zeta) \, dv = \iint\limits_S \left[\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \oint (\xi^\alpha, \zeta) \, \mu \, d\zeta \right] da \quad , \qquad (4.3)$$

which is the desired integral representation. A similar argument yields the representation
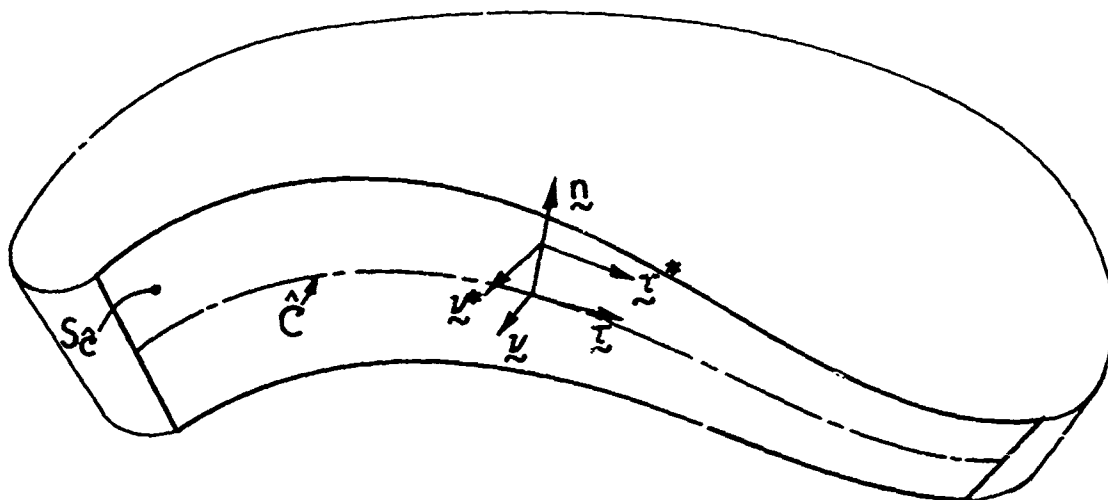
Figure 4.2 The surface $S_{\hat{c}}$ generated by varying ζ between the limits $\pm\frac{h}{2}$ over the arc $\hat{C}$. τ is the unit tangent to $\hat{C}$ and ν the exterior unit normal to $\hat{C}$ tangent to the middle surface. ν^* is normal to $S_{\hat{c}}$ and τ^* is tangent to $S_{\hat{c}}$ and the middle surface, but ν^* and τ^* need not be unit vectors.

$$\iint\limits_{S_{\pm}} \oint (\xi^{\alpha})\, da = \iint\limits_{S} \oint (\xi^{\alpha})\, \mu_{\pm}\, da^* \tag{4.4}$$

for any function $\oint(\xi^{\alpha})$ defined on either S_+ or S_- , with μ_+ and μ_- the values of μ at $\zeta = \frac{h}{2}$ and $\zeta = -\frac{h}{2}$ respectively.

We now seek an expression relating an integral over the bounding surface S_c to an integral along the curve C of an integral through the thickness. We begin by taking some connected subset of C, say the arc $\hat{C}$, and the surface $S_{\hat{c}}$ generated from $\hat{C}$ by varying $-\frac{h}{2} \leq \zeta \leq \frac{h}{2}$, as shown in Figure 4.2. Let $\oint (s, \zeta)$ be a function defined on $S_{\hat{c}}$, with s the arc length along $\hat{C}$. The integral $\oint (s, \zeta)$ over $S_{\hat{c}}$ can be expressed as

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*This expression is exact only when h = const. However, for slowly varying h it is a good approximation.

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$$\iint\limits_{S_{\hat{C}}} \oint (s,\zeta) \, da = \int\limits_{\hat{C}} \left[\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \oint (s,\zeta) \mid \underset{\sim}{v}^* \mid d\zeta \right] ds^*, \qquad (4.5)$$

where $\underset{\sim}{v}^*$ is the (not necessarily unit) outward normal to $S_{\hat{C}}$ given by the expression

$$\underset{\sim}{v}^* = \underset{\sim}{\tau}^* \times \underset{\sim}{n} \quad , \qquad (4.6)$$

with

$$\underset{\sim}{\tau}^* \equiv \frac{\partial \underset{\sim}{x}}{\partial s} = \underset{\sim}{g}_\alpha \frac{d\xi^\alpha}{ds} \equiv \underset{\sim}{g}_\alpha \, \tau^\alpha \qquad (4.7)$$

by virtue of (2.36). It should be carefully noted that τ^α denotes the components of the unit tangent τ to $\hat{C}$ with respect to the basis $\underset{\sim}{a}_\alpha$. It can be shown, using $(2.14)_3$, $(2.46)_4$ and $(2.47)_2$ that

$$\underset{\sim}{g}_\alpha \times \underset{\sim}{n} = - \mu \, \epsilon_{\alpha\beta} \, \underset{\sim}{g}^\beta \quad , \qquad (4.8)$$

which when used in conjunction with (4.7) enables us to write $\underset{\sim}{v}^*$ as

$$\underset{\sim}{v}^* = \mu \, \epsilon_{\beta\alpha} \, \tau^\alpha \, \underset{\sim}{g}^\beta \qquad , \qquad (4.9)$$

or, in view of $(A.9)_2$, as

$$\underset{\sim}{v}^* = \mu \, \nu_\beta \, \underset{\sim}{g}^\beta \quad . \qquad (4.10)$$

The reader should notice that ν_β represents the components with respect to $\underset{\sim}{a}_\alpha$ of the unit outward normal to $S_{\hat{C}}$ along $\hat{C}$. The last expression could be substituted in (4.5), but we refrain from this since no simplification results at the moment.

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*See SOKOLNIKOFF [7; p. 151].

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4.2 Linear Momentum and Spin Momentum Relations

For the following derivation we add to the Kirchhoff hypothesis the two basic assumptions: 1° the momentum density of particles composing the shell is the product of the mass density and particle velocity and 2° the linear momentum and moment of momentum acting on any portion of the shell are the resultants of the linear momentum and moment of momentum densities of the particles composing that part of the shell.

We begin by endowing the middle surface with inertia by ascribing to each point composing the surface a *linear momentum* P per unit area and a *spin momentum* ℓ per unit area. Using the notation of the last section, we have as a consequence of applying the above assumptions to the volume V the two integral equations

$$\iint_S P \, da = \iiint_V \rho \, \dot{x} \, dv \ , \quad \iint_S (r \times P + \ell) \, da = \iiint_V x \times (\rho \, \dot{x}) \, dv \ , \quad (4.11)$$

with ρ the mass per unit volume and $\dot{x}$ the particle velocity. Upon applying the integral representation (4.3), these become

$$\iint_S P \, da = \iint_S \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \, \dot{x} \, \mu \, d\zeta \right] da \ ,$$

$$\tag{4.12}$$

$$\iint_S (r \times P + \ell) \, da = \iint_S \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \, x \times \dot{x} \, \mu \, d\zeta \right] da \ .$$

Since S is arbitrary, it follows from the continuity of ρ and $\dot{x}$ that the integrands are equal:

$$P = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \, \dot{x} \, \mu \, d\zeta \ , \quad r \times P + \ell = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \, x \times \dot{x} \, \mu \, d\zeta \ . \quad (4.13)$$

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By virtue of the Kirchhoff hypothesis, (2.34) and (2.35), the last equations become the *momentum-velocity relations*

$$P = \gamma_0 \, \underset{\sim}{v} + \gamma_1 \, \underset{\sim}{\omega} \quad , \qquad \underset{\sim}{\ell} = \underset{\sim}{n} \times (\gamma_1 \, \underset{\sim}{v} + \gamma_2 \, \underset{\sim}{\omega}) \quad , \qquad (4.14)$$

where γ_0, γ_1, γ_2 are the zeroth, first and second resultant moments of mass density, defined by the general expression

$$\gamma_a = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \, \mu \, (\zeta)^a \, d\zeta \qquad (a = 0, \, 1, \, 2) \quad . \qquad (4.15)$$

These resultants have physical interpretations: γ_0 is the mass per unit middle surface area, γ_1 is the product of the mass per unit middle surface area and the distance from the middle surface to the center of mass, and γ_2 is the moment of inertia of the section relative to the middle surface. If the center of mass were to coincide with the middle surface, then γ_1 would vanish and γ_2 would be minimum.

4.3 Stress Resultant and Couple Resultant Relations

Since we assume the material to be nonpolar, no couple stress field is defined over the shell and only a (force) stress field need be considered. The force and torque acting on any finite surface element are assumed to be the resultants of the stress and the moment of stress acting on the surface element.

Let C be any closed curve in the middle surface, then we assume that the action of the surface exterior to C on the part interior to C is equivalent to a *stress resultant vector* $Q_{(\nu)}$ per unit arc length and a *couple resultant vector* $\underset{\sim}{m}_{(\nu)}$ per unit arc length.[*] It is to be

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[*] This is a statement of the stress principle for shells postulated by ERICKSEN & TRUESDELL [11].

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expected that these resultant vectors will be functions not only of the point through which C passes but also of the curve C itself. We anticipate that the dependence on C will be through the exterior unit normal ν to the curve tangential to the surface by writing the stress resultant and couple resultant vectors with subscript (ν).

Using the notation introduced in 4.1 and applying the above assumptions to the surface $S_{\hat{C}}$ generated by the arc $\hat{C}$ subset of the closed curve C (see Figures 4.1 and 4.2), we obtain

$$\int_{\hat{C}} Q_{(\nu)} \, ds = \iint_{S_{\hat{C}}} \sigma \cdot \hat{\nu} \, da \quad ,$$

$$\int_{\hat{C}} (r \times Q_{(\nu)} + m_{(\nu)}) \, ds = \iint_{S_{\hat{C}}} x \times \sigma \cdot \hat{\nu} \, da \quad ,$$

(4.16)

with σ the stress tensor and $\hat{\nu}$ the unit exterior normal to $S_{\hat{C}}$, so that $\sigma \cdot \hat{\nu}$ is the force per unit area over $S_{\hat{C}}$. Since $\hat{\nu}$ is a unit vector co-linear with ν^*, we can write

$$\nu^* = |\nu^*| \, \hat{\nu} \quad , \tag{4.17}$$

so that on applying the representation (4.5) to the above integrals they become

$$\int_{\hat{C}} Q_{(\nu)} \, ds = \int_{\hat{C}} \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma \cdot \nu^* \, d\zeta \right] ds \quad ,$$

(4.18)

$$\int_{\hat{C}} (r \times Q_{(\nu)} + m_{(\nu)}) \, ds = \int_{\hat{C}} \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} x \times \sigma \cdot \nu^* \, d\zeta \right] ds \quad .$$

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Since these integrals hold for all arcs $\hat{C}$ subsets of C, it follows from the continuity of the stress tensor that the integrands are equal, so that in view of (2.34) we obtain

$$\underset{\sim}{Q}_{(\nu)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \underset{\sim}{\sigma} \cdot \underset{\sim}{\nu}^* \, d\zeta \quad , \qquad \underset{\sim}{m}_{(\nu)} = \underset{\sim}{n} \times \int_{-\frac{h}{2}}^{\frac{h}{2}} \underset{\sim}{\sigma} \cdot \underset{\sim}{\nu}^* \, \zeta \, d\zeta \quad . \quad (4.19)$$

Next we use (4.10) enabling us to write $\underset{\sim}{Q}_{(\nu)}$ and $\underset{\sim}{m}_{(\nu)}$ as linear functions of the components of the exterior unit normal $\underset{\sim}{\nu}$ to the arc $\hat{C}$:

$$\underset{\sim}{Q}_{(\nu)} = \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \underset{\sim}{\sigma} \cdot \underset{\sim}{g}^{\alpha} \, \mu \, d\zeta \right] \nu_{\alpha} \quad ,$$

$$\underset{\sim}{m}_{(\nu)} = \left[\underset{\sim}{n} \times \int_{-\frac{h}{2}}^{\frac{h}{2}} \underset{\sim}{\sigma} \cdot \underset{\sim}{g}^{\alpha} \, \mu \, \zeta \, d\zeta \right] \nu_{\alpha} \quad . \tag{4.20}$$

Or defining the *stress resultant tensor* $\underset{\sim}{Q}^{\alpha}$ and *couple resultant tensor*[*] $\underset{\sim}{m}^{\alpha}$ as follows

$$\underset{\sim}{Q}^{\alpha} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \underset{\sim}{\sigma} \cdot \underset{\sim}{g}^{\alpha} \, \mu \, d\zeta \quad , \qquad \underset{\sim}{m}^{\alpha} = \underset{\sim}{n} \times \int_{-\frac{h}{2}}^{\frac{h}{2}} \underset{\sim}{\sigma} \cdot \underset{\sim}{g}^{\alpha} \, \mu \, \zeta \, d\zeta \quad , \quad (4.21)$$

(notice that both are independent of the curve normal $\underset{\sim}{\nu}$), we can finally

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[*] $\underset{\sim}{Q}^{\alpha}$ and $\underset{\sim}{m}^{\alpha}$ are called tensors because each is pair of vectors in 3-space that, in addition, behave like components of vectors in the (2-dimensional) tangent space of the middle surface under transformation of the material coordinates ξ^{α}. More properly they should be called double tensor fields, see [11; Eq. 3.1] for a definition.

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write the stress and couple resultant vectors as

$$\underset{\sim}{Q}_{(\nu)} = \underset{\sim}{Q}^{\alpha} \nu_{\alpha} \quad , \qquad \underset{\sim}{m}_{(\nu)} = \underset{\sim}{m}^{\alpha} \nu_{\alpha} \quad^{*} \quad . \tag{4.22}$$

Because $\underset{\sim}{m}^{\alpha}$ lacks a normal component, as is clear from $(4.21)_2$, we find it convenient to introduce an equivalent tensor $\underset{\sim}{M}^{\alpha}$, the *bending resultant tensor*, through the definition

$$\underset{\sim}{M}^{\alpha} = \underset{\sim}{m}^{\alpha} \times \underset{\sim}{n} \quad , \tag{4.23}$$

with the equivalence of these two tensors following from the conjugate relation

$$\underset{\sim}{m}^{\alpha} = \underset{\sim}{n} \times \underset{\sim}{M}^{\alpha} \quad , \tag{4.24}$$

which is a consequence of $(4.21)_2$. Like $\underset{\sim}{m}^{\alpha}$, $\underset{\sim}{M}^{\alpha}$ only has tangential components.

The stress resultant $\underset{\sim}{Q}^{\alpha}$ and bending resultant $\underset{\sim}{M}^{\alpha}$ are conveniently resolved relative to the basis $(\underset{\sim}{a}_{\alpha}, \underset{\sim}{n})$:

$$\underset{\sim}{Q}^{\alpha} = Q^{\alpha\beta} \underset{\sim}{a}_{\beta} + Q^{\alpha} \underset{\sim}{n} \quad , \qquad \underset{\sim}{M}^{\alpha} = M^{\alpha\beta} \underset{\sim}{a}_{\beta} \quad , \tag{4.25}$$

and, by virtue of (4.21) and (4.23), we obtain for their components

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* These relations could also have been obtained from integral equations of equilibrium by arguments based on the continuity of the integrands and the arbitrariness of S, c f. ERICKSEN & TRUESDELL [11; Sect. 24].

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$$Q^{\alpha\beta} = \underset{\sim}{a}^{\beta} \cdot \int_{-\frac{h}{2}}^{\frac{h}{2}} \underset{\sim}{\sigma} \cdot \underset{\sim}{g}^{\alpha} \, \mu \, d\zeta \, , \qquad Q^{\alpha} = \underset{\sim}{n} \cdot \int_{-\frac{h}{2}}^{\frac{h}{2}} \underset{\sim}{\sigma} \cdot \underset{\sim}{g}^{\alpha} \, \mu \, d\zeta \, ,$$

$$(4.26)$$

$$M^{\alpha\beta} = \underset{\sim}{a}^{\beta} \cdot \int_{-\frac{h}{2}}^{\frac{h}{2}} \underset{\sim}{\sigma} \cdot \underset{\sim}{g}^{\alpha} \, \mu \, \zeta \, d\zeta \, .$$

Writing the components of the stress $\underset{\sim}{\sigma}$ relative to the basis $\underset{\sim}{g}_i$ as follows

$$\sigma^{ij} = \underset{\sim}{g}^{i} \cdot \underset{\sim}{\sigma} \cdot \underset{\sim}{g}^{j} \quad , \qquad (4.27)$$

we can, using $(2.45)_3$ and $(2.46)_4$, replace $\underset{\sim}{\sigma}$ by its components to obtain

$$Q^{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \mu^{\beta}_{\gamma} \, \sigma^{\gamma\alpha} \, \mu \, d\zeta \, , \qquad Q^{\alpha} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{3\alpha} \, \mu \, d\zeta \, ,$$

$$(4.28)$$

$$M^{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \mu^{\beta}_{\gamma} \, \sigma^{\gamma\alpha} \, \mu \, \zeta \, d\zeta \, .$$

By convention we call $Q^{\alpha\beta}$ the *membrane components* and Q^{α} the *shear components* of the stress resultant. The last relations, as well as the one before, will hold irrespective of how the stress is obtained from the strain and/or strain rates, etc. Although not used in this derivation, the stress is assumed to be symmetric

$$\sigma^{ij} = \sigma^{ji} \quad , \qquad (4.29)$$

in line with the nonpolar nature of the material.

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4.4 External Force and External Couple Relations

Again, using the fact that the material is nonpolar we consider only a body force field defined over the shell. We assume that the force on a finite volume is the resultant of the stress acting over the volume's boundary and the body force acting over the volume, and that the torque on the volume is the resultant of the moment of stress over the boundary and the moment of body force over the volume.

We begin by accounting for the influence of the external environment on the middle surface by assigning to each point on the surface an *external force* F per unit area and an *external couple* c per unit area. We then apply the above assumptions to the volume V bounded by the surfaces S_c, S_+ and S_- (see Figure 4.1) to obtain, after taking care to eliminate the contributions from S_c by applying (4.16) to C, the two integral equations

$$\iint_S F \, da = \iiint_V b \, dv + \iint_{S_+} t_+ \, da + \iint_{S_-} t_- \, da \quad ,$$

$$(4.30)$$

$$\iint_S (r \times F + c) \, da = \iiint_V x \times b \, dv + \iint_{S_+} x_+ \times t_+ \, da + \iint_{S_-} x_- \times t_- \, da \quad ,$$

with b the body force, and t_+ and t_- the tractions on the surfaces where $\zeta = \frac{h}{2}$ and $\zeta = -\frac{h}{2}$, respectively. Since the exterior unit normals to S_+ and S_- are n and $-n$, we obtain for the tractions

$$t_+ = \sigma_+ \cdot n \quad , \quad t_- = -\sigma_- \cdot n \quad . \quad (4.31)$$

When we apply (4.3) and (4.4) to the last integrals, they become

$$\iint\limits_{S} \underset{\sim}{F} \, da = \iint\limits_{S} \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \underset{\sim}{b} \, \mu \, d\zeta + \underset{\sim}{t}_+ \, \mu_+ + \underset{\sim}{t}_- \, \mu_- \right] da \quad ,$$

(4.32)

$$\iint\limits_{S} (\underset{\sim}{r} \times \underset{\sim}{F} + \underset{\sim}{c}) \, da = \iint\limits_{S} \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \underset{\sim}{x} \times \underset{\sim}{b} \, \mu \, d\zeta + \underset{\sim}{x}_+ \times \underset{\sim}{t}_+ \, \mu_+ + \underset{\sim}{x}_- \times \underset{\sim}{t}_- \, \mu_- \right] da.$$

From the arbitrariness of S and the continuity of the body force and tractions it follows that the integrands are equal. Consequently, in view of the Kirchhoff hypothesis (2.34), we have

$$\underset{\sim}{F} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \underset{\sim}{b} \, \mu \, d\zeta + \underset{\sim}{t}_+ \, \mu_+ + \underset{\sim}{t}_- \, \mu_- \quad ,$$

(4.33)

$$\underset{\sim}{c} = \underset{\sim}{n} \times \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \underset{\sim}{b} \, \mu \, \zeta \, d\zeta + \frac{h}{2} \, (\underset{\sim}{t}_+ \, \mu_+ - \underset{\sim}{t}_- \, \mu_-) \right] \quad ,$$

where it should be noticed that $\underset{\sim}{c}$ is tangential. For convenience, we prefer to work with the equivalent tangential vector $\underset{\sim}{C}$ rather than $\underset{\sim}{c}$, which satisfies the relations

$$\underset{\sim}{C} = \underset{\sim}{c} \times \underset{\sim}{n} \quad , \qquad \underset{\sim}{c} = \underset{\sim}{n} \times \underset{\sim}{C} \quad . \tag{4.34}$$

Resolving $\underset{\sim}{F}$ and $\underset{\sim}{C}$ relative to the basis $(\underset{\sim}{a}_\alpha, \underset{\sim}{n})$:

$$\underset{\sim}{F} = F^\alpha \, \underset{\sim}{a}_\alpha + F \, \underset{\sim}{n} \quad , \qquad \underset{\sim}{C} = C^\alpha \, \underset{\sim}{a}_\alpha \quad , \tag{4.35}$$

we determine their components from (4.33)

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$$F^\alpha = \int_{-\frac{h}{2}}^{\frac{h}{2}} \mu_\beta^\alpha \; b^\beta \; \mu \; d\zeta + \mu_{+\beta}^\alpha \; t_+^\beta \; \mu_+ + \mu_{-\beta}^\alpha \; t_-^\beta \; \mu_- \quad ,$$

$$F = \int_{-\frac{h}{2}}^{\frac{h}{2}} b^3 \; \mu \; d\zeta + t_+^3 \; \mu_+ + t_-^3 \; \mu_- \quad , \tag{4.36}$$

$$C^\alpha = \int_{-\frac{h}{2}}^{\frac{h}{2}} \mu_\beta^\alpha \; b^\beta \; \mu \; \zeta \; d\zeta + \frac{h}{2} (\mu_{+\beta}^\alpha \; t_+^\beta \; \mu_+ - \mu_{-\beta}^\alpha \; t_-^\beta \; \mu_-) \quad ,$$

wherein the components of $\underset{\sim}{b}$ and $\underset{\sim}{t}$ are with respect to the basis $\underset{\sim}{g}_i$:

$$b^i = \underset{\sim}{g}^i \cdot \underset{\sim}{b} \quad , \qquad t^i = \underset{\sim}{g}^i \cdot \underset{\sim}{t} \quad . \tag{4.37}$$

5. EQUATIONS OF MOTION OF A SHELL

In this chapter we first derive general equations of motion for a shell, with no assumptions (such as the Kirchhoff hypothesis) being made about the deformation through the thickness. Next the consequences of mass being conserved on the resultant moments of mass density are found. These are then combined with the resultant relations of the last chapter to simplify the equations of motion. Lastly, the SANDERS-LEONARD procedure is used to eliminate the shear components of the stress resultant, resulting in a single vector equation of motion.

5.1 Derivation of Equations of Motion

The derivation is based on formulating the principles of balance of linear momentum and moment of momentum for the reference surface in integral form. From the last chapter we have that the reference surface[*] is endowed with linear momentum P and spin momentum ℓ per unit surface area. Also the mechanical influence of the external environment on the reference surface is specified through the external force F and external couple c per unit area.[**]

Lastly, we have from the previous chapter that for any simple closed curve in the reference surface, the mechanical influence of the part of the surface exterior to the curve on the part interior is transmitted through the stress resultant vector $Q_{(\nu)}$ and the couple resultant vector $m_{(\nu)}$ per unit arc length. Using these variables, we can state the principles of balance of linear momentum and moment of momentum for any simply connected region S of the reference surface with simple closed boundary C (see Figure 5.1) in the following integral forms:

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[*] We use reference surface, rather than middle surface, to emphasize that the results of this section are independent of the Kirchhoff hypothesis.

[**] For completeness, we should introduce here the contact force Q and contact couple m per unit arc length to account for the influence of the environment entering through the physical boundaries, but we postpone this until the next chapter, since these quantities are not used here.
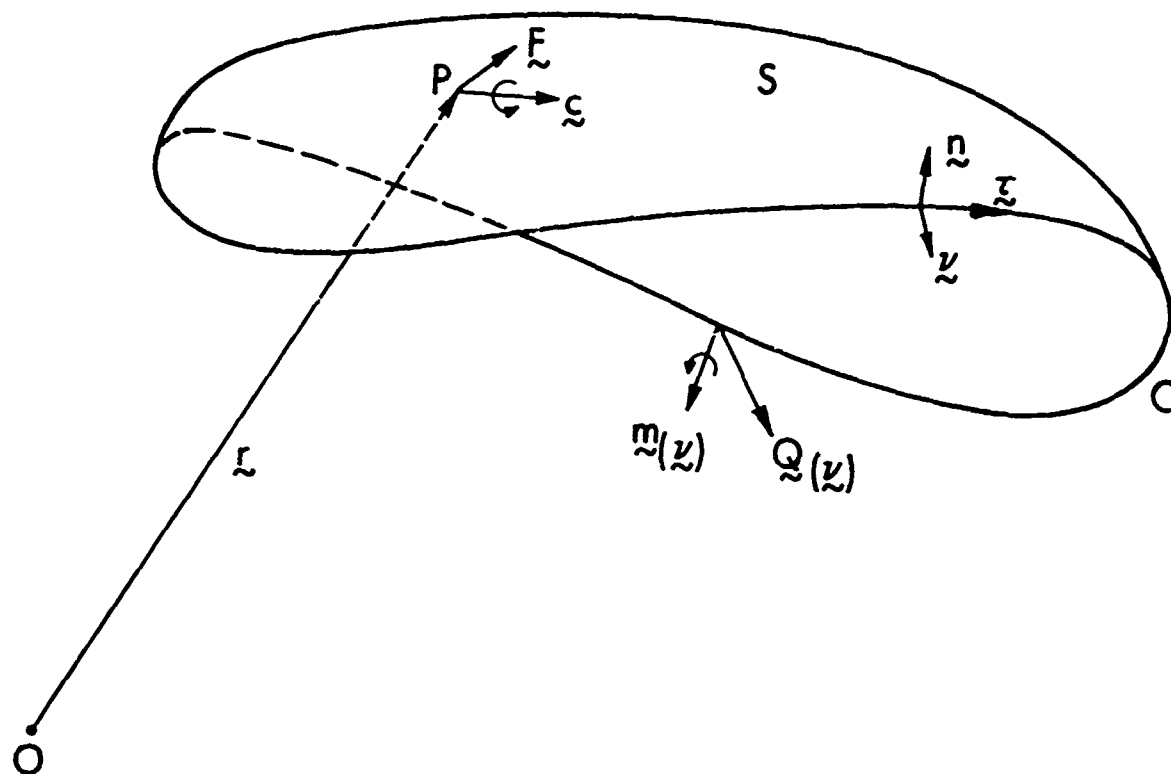
Figure 5.1 The action on the region S of the external force $\underset{\sim}{F}$, external couple $\underset{\sim}{c}$, stress resultant vector $\underset{\sim}{Q}_{(\nu)}$ and couple resultant vector $\underset{\sim}{m}_{(\nu)}$. $\underset{\sim}{F}$ and $\underset{\sim}{c}$ are shown acting on ⸱ typical point p in S located by position vector $\underset{\sim}{r}$, while $\underset{\sim}{Q}_{(\nu)}$ and $\underset{\sim}{m}_{(\nu)}$ are shown acting on a typical point on the boundary C of the surface S. Also indicated is the relative orientation along C of the surface unit normal $\underset{\sim}{n}$, the boundary unit tangent $\underset{\sim}{\tau}$ and external unit normal $\underset{\sim}{\nu}$.

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$$\frac{d}{dt} \iint_S \underset{\sim}{P}\, da = \oint_C \underset{\sim}{Q}_{(\nu)}\, ds + \iint_S \underset{\sim}{F}\, da \quad,$$

(5.1)

$$\frac{d}{dt} \iint_S (\underset{\sim}{r} \times \underset{\sim}{P} + \underset{\sim}{\ell})\, da = \oint_C (\underset{\sim}{r} \times \underset{\sim}{Q}_{(\nu)} + \underset{\sim}{m}_{(\nu)})\, ds + \iint_S (\underset{\sim}{r} \times \underset{\sim}{F} + \underset{\sim}{c})\, da \;.$$

From these integral relations we derive the differential equations of motion. Assuming that the inertia terms in the left hand integrals are continuous, we can apply Reynold's transport theorem[*] to bring the material derivative past the integral signs:

$$\iint_S (\dot{\underset{\sim}{P}} + \nabla \cdot \underset{\sim}{v}\, \underset{\sim}{P})\, da = \oint_C \underset{\sim}{Q}_{(\nu)}\, ds + \iint_S \underset{\sim}{F}\, da \quad,$$

$$\iint_S \left[\dot{\underset{\sim}{\ell}} + \nabla \cdot \underset{\sim}{v}\, \underset{\sim}{\ell} + \underset{\sim}{v} \times \underset{\sim}{P} + \underset{\sim}{r} \times (\dot{\underset{\sim}{P}} + \nabla \cdot \underset{\sim}{v}\, \underset{\sim}{P}) \right] da =$$

(5.2)

$$\oint_C (\underset{\sim}{r} \times \underset{\sim}{Q}_{(\nu)} + \underset{\sim}{m}_{(\nu)})\, ds + \iint_S (\underset{\sim}{r} \times \underset{\sim}{F} + \underset{\sim}{c})\, da \;.$$

As noted earlier,[**] equations (4.22) relating $\underset{\sim}{Q}_{(\nu)}$ and $\underset{\sim}{m}_{(\nu)}$ to tensors $\underset{\sim}{Q}^{\alpha}$ and $\underset{\sim}{m}^{\alpha}$ can be obtained directly from the balance equations (5.1) without recourse to the special material and geometric assumptions of the last chapter. Assuming only that the surface integrands are bounded and the line integrands continuous, the usual limit argument using a

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[*] A derivation of Reynold's transport theorem for a surface is included in Appendix A.

[**] See footnote on page 47.

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sequence of regions in S gives (4.22), but *without* the restriction that $\underset{\sim}{m}^{\alpha}$ be tangential. Substitution of (4.2?) in the above integrals allows us to use the divergence theorem[*] to replace the line integrals by surface integrals:

$$\iint_S (\dot{\underset{\sim}{P}} + \nabla \cdot \underset{\sim}{v}\, \underset{\sim}{P})\, da = \iint_S (\underset{\sim}{Q}^{\alpha}{}_{|\alpha} + \underset{\sim}{F})\, da \quad ,$$

$$\iint_S \left[\dot{\underset{\sim}{\ell}} + \nabla \cdot \underset{\sim}{v}\, \underset{\sim}{\ell} + \underset{\sim}{v} \times \underset{\sim}{P} + \underset{\sim}{r} \times (\dot{\underset{\sim}{P}} + \nabla \cdot \underset{\sim}{v}\, \underset{\sim}{P}) \right]\, da = \qquad (5.3)$$

$$\iint_S \left[\underset{\sim}{m}^{\alpha}{}_{|\alpha} + \underset{\sim}{a}_{\alpha} \times \underset{\sim}{Q}^{\alpha} + \underset{\sim}{c} + \underset{\sim}{r} \times (\underset{\sim}{Q}^{\alpha}{}_{|\alpha} + \underset{\sim}{F}) \right]\, da \quad .$$

Finally, assuming the above integrands continuous, then by virtue of S being arbitrary it follows that (5.3) can be satisfied if and only if

$$\dot{\underset{\sim}{P}} + \nabla \cdot \underset{\sim}{v}\, \underset{\sim}{P} = \underset{\sim}{Q}^{\alpha}{}_{|\alpha} + \underset{\sim}{F} \quad ,$$

$$(5.4)$$

$$\dot{\underset{\sim}{\ell}} + \nabla \cdot \underset{\sim}{v}\, \underset{\sim}{\ell} + \underset{\sim}{v} \times \underset{\sim}{P} = \underset{\sim}{m}^{\alpha}{}_{|\alpha} + \underset{\sim}{a}_{\alpha} \times \underset{\sim}{Q}^{\alpha} + \underset{\sim}{c} \quad .$$

These are the *differential equations of motion* for a shell, the first embodying the principle of balance of linear momentum and the second the principle of balance of moment of momentum. Under the assumed continuity conditions, they are fully equivalent to (5.1). They are independent of the properties of the material composing the shell

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[*]A derivation of the divergence theorem for a surface is included in Appendix A.

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and of the Kirchhoff hypothesis, with no restrictions placed on $\underset{\sim}{m}^\alpha$ and $\underset{\sim}{c}$ being tangential. Setting the inertia terms on the left hand sides equal to zero, we get the *differential equations of equilibrium* for a shell:

$$\underset{\sim}{Q}^\alpha{}_{|\alpha} + \underset{\sim}{F} = 0 \quad , \qquad \underset{\sim}{m}^\alpha{}_{|\alpha} + \underset{\sim}{a}_\alpha \times \underset{\sim}{Q}^\alpha + \underset{\sim}{c} = 0 \quad . \qquad (5.5)$$

They are the vectorial forms of the equations derived by ERICKSEN & TRUESDELL [11; Eq. 25.2].

Next we write the equations of motion in terms of components normal and tangential to the reference surface. We begin by resolving the variables in (5.4) into normal and tangential components:

$$\underset{\sim}{P} = P^\beta \underset{\sim}{a}_\beta + P \underset{\sim}{n} \quad , \qquad \underset{\sim}{\ell} = \ell^\beta \underset{\sim}{a}_\beta + \ell \underset{\sim}{n} \quad ,$$

$$\underset{\sim}{Q}^\alpha = Q^{\alpha\beta} \underset{\sim}{a}_\beta + Q^\alpha \underset{\sim}{n} \quad , \qquad \underset{\sim}{m}^\alpha = m^{\alpha\beta} \underset{\sim}{a}_\beta + m^\alpha \underset{\sim}{n} \quad , \qquad (5.6)$$

$$\underset{\sim}{F} = F^\beta \underset{\sim}{a}_\beta + F \underset{\sim}{n} \quad , \qquad \underset{\sim}{c} = c^\beta \underset{\sim}{a}_\beta + c \underset{\sim}{n} \quad .$$

By virtue of (2.26), (2.29) and (2.33), it follows that

$$\dot{\underset{\sim}{P}} = [\dot{P}^\beta + P^\gamma (v^\beta{}_{|\gamma} - b^\beta_\gamma v) + P \omega^\beta] \underset{\sim}{a}_\beta + [\dot{P} - P^\beta \omega_\beta] \underset{\sim}{n} \quad ,$$

$$(5.7)$$

$$\dot{\underset{\sim}{\ell}} = [\dot{\ell}^\beta + \ell^\gamma (v^\beta{}_{|\gamma} - b^\beta_\gamma v) + \ell \omega^\beta] \underset{\sim}{a}_\beta + [\dot{\ell} - \ell^\beta \omega_\beta] \underset{\sim}{n} \quad .$$

As a consequence of (2.20), we have

$$\underset{\sim}{Q}{}^{\alpha}{}_{|\alpha} = [Q^{\alpha\beta}{}_{|\alpha} - b^{\beta}_{\alpha} Q^{\alpha}] \underset{\sim}{a}_{\beta} + [Q^{\alpha}{}_{|\alpha} + b_{\alpha\beta} Q^{\alpha\beta}] \underset{\sim}{n} \quad ,$$

$$\underset{\sim}{m}{}^{\alpha}{}_{|\alpha} = [m^{\alpha\beta}{}_{|\alpha} - b^{\beta}_{\alpha} m^{\alpha}] \underset{\sim}{a}_{\beta} + [m^{\alpha}{}_{|\alpha} + b_{\alpha\beta} m^{\alpha\beta}] \underset{\sim}{n} \quad .$$

$$(5.8)$$

The cross product terms in (5.4) are resolved by using (2.14):

$$\underset{\sim}{v} \times \underset{\sim}{P} = \varepsilon_{\alpha\beta} (v P^{\alpha} - v^{\alpha} P) \underset{\sim}{a}^{\beta} + \varepsilon_{\alpha\beta} v^{\alpha} P^{\beta} \underset{\sim}{n} \quad ,$$

$$\underset{\sim}{a}_{\alpha} \times Q^{\alpha} = - \varepsilon_{\alpha\beta} Q^{\alpha} \underset{\sim}{a}^{\beta} + \varepsilon_{\alpha\beta} Q^{\alpha\beta} \underset{\sim}{n} \quad .$$

$$(5.9)$$

Substitution of the last three results and (2.27) into the equations of motion (5.4) yields

$$\dot{P}^{\beta} + P^{\alpha} (v^{\beta}{}_{|\alpha} - b^{\beta}_{\alpha} v) + P^{\beta} (v^{\alpha}{}_{|\alpha} - b^{\alpha}_{\alpha} v) + P \omega^{\beta} = Q^{\alpha\beta}{}_{|\alpha} - b^{\beta}_{\alpha} Q^{\alpha} + F^{\beta} \quad ,$$

$$\dot{P} + P (v^{\alpha}{}_{|\alpha} - b^{\alpha}_{\alpha} v) - P^{\alpha} \omega_{\alpha} = Q^{\alpha}{}_{|\alpha} + b_{\alpha\beta} Q^{\alpha\beta} + F \quad ,$$

$$(5.10)$$

$$\dot{\ell}^{\beta} + \ell^{\alpha} (v^{\beta}{}_{|\alpha} - b^{\beta}_{\alpha} v) + \ell^{\beta} (v^{\alpha}{}_{|\alpha} - b^{\alpha}_{\alpha} v) + \ell \omega^{\beta} + \varepsilon_{\alpha}{}^{\beta} (v P^{\alpha} - v^{\alpha} P)$$

$$= m^{\alpha\beta}{}_{|\alpha} - b^{\beta}_{\alpha} m^{\alpha} - \varepsilon_{\alpha}{}^{\beta} Q^{\alpha} + c^{\beta} \quad ,$$

$$\dot{\ell} + \ell (v^{\alpha}{}_{|\alpha} - b^{\alpha}_{\alpha} v) - \ell^{\alpha} \omega_{\alpha} + \varepsilon_{\alpha\beta} v^{\alpha} P^{\beta} = m^{\alpha}{}_{|\alpha} + b_{\alpha\beta} m^{\alpha\beta} + \varepsilon_{\alpha\beta} Q^{\alpha\beta} + c \quad ,$$

the *tangential and normal components of the equations of motion* for a

shell. The first and second equations embody the principle of balance of linear momentum and the third and fourth the principle of balance of moment of momentum. On setting the inertia terms equal to zero, the component form of the equations of equilibrium result, coinciding exactly with those derived by ERICKSEN & TRUESDELL [11; Eq. (26.6 - 26.9)].

An alternative formulation of the equations of motion (5.4), in which the covariant derivatives are replaced by partial derivatives and the velocity divergence terms vanish, can be obtained by utilizing the relations (2.18) and (2.28). Defining the new variables:

$$\underset{\sim}{P}{}^* = a^{\frac{1}{2}} \underset{\sim}{P} \quad , \qquad \underset{\sim}{Q}{}^{*\alpha} = a^{\frac{1}{2}} \underset{\sim}{Q}{}^{\alpha} \quad , \qquad \underset{\sim}{F}{}^* = a^{\frac{1}{2}} \underset{\sim}{F} \quad ,$$

$$(5.11)$$

$$\underset{\sim}{\ell}{}^* = a^{\frac{1}{2}} \underset{\sim}{\ell} \quad , \qquad \underset{\sim}{m}{}^{*\alpha} = a^{\frac{1}{2}} \underset{\sim}{m}{}^{\alpha} \quad , \qquad \underset{\sim}{c}{}^* = a^{\frac{1}{2}} \underset{\sim}{c} \quad ,$$

it follows by virtue of these relations that the equations of motion become:

$$\underset{\sim}{\dot{P}}{}^* = \underset{\sim}{Q}{}^{*\alpha}{}_{,\alpha} + \underset{\sim}{F}{}^* \quad , \qquad \underset{\sim}{\dot{\ell}}{}^* + \underset{\sim}{v} \times \underset{\sim}{P}{}^* = \underset{\sim}{m}{}^{*\alpha}{}_{,\alpha} + \underset{\sim}{a}{}_{\alpha} \times \underset{\sim}{Q}{}^{*\alpha} + \underset{\sim}{c}{}^* \quad . \quad (5.12)$$

All equations derived in this section, as mentioned before, are completely general and do not depend on the form assumed for the deformation through the shell thickness. Hence, these equations are valid for all shell theories.

5.2 Conservation of Mass

We now derive the consequences of mass being conserved on the resultant moments of mass density γ_0, γ_1, γ_2. As a result of the mass being conserved for any portion of the shell, the mass density ρ satisfies the well known equation of mass continuity

$$\dot{\rho} + \rho \operatorname{Div} \underset{\sim}{\dot{x}} = 0 \quad . \tag{5.13}$$

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Writing the divergence in the (ξ^α, ζ) coordinate system:

$$\text{Div } \dot{\underset{\sim}{x}} = \underset{\sim}{g}^\alpha \cdot \dot{\underset{\sim}{x}},_\alpha + \underset{\sim}{g}^3 \cdot \frac{\partial \underset{\sim}{x}}{\partial \zeta} \quad ,^* \qquad (5.14)$$

we obtain by virtue of (2.29), (2.36) and (2.45)$_3$

$$\text{Div } \dot{\underset{\sim}{x}} = \underset{\sim}{g}^\alpha \cdot \dot{\underset{\sim}{g}}_\alpha + \underset{\sim}{n} \cdot \underset{\sim}{\omega} \quad . \qquad (5.15)$$

Since $\underset{\sim}{\omega}$ is tangential, the second right hand term vanishes and using (2.45)$_2$ we have

$$\text{Div } \dot{\underset{\sim}{x}} = \frac{1}{2} g^{\alpha\beta} \dot{g}_{\alpha\beta} \equiv \dot{g}^{\frac{1}{2}} / g^{\frac{1}{2}} \quad . \qquad (5.16)$$

Applying (2.46)$_3$ to this result, the equation of mass continuity becomes

$$\frac{d}{dt} (\rho \, u \, a^{\frac{1}{2}}) = 0 \quad . \qquad (5.17)$$

Consequently, the product $\rho \, u \, a^{\frac{1}{2}}$ is a time constant and hence equal to its initial value at time $t = t_0$:

$$\rho \, u \, a^{\frac{1}{2}} = \rho_0 \, u_0 \, A^{\frac{1}{2}} \, .^{**} \qquad (5.18)$$

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*See GREEN & ZERNA [12; Eq. 1.12.36].

**See the footnote on page 31.

Applying this result to the defining equations for γ_a (4.15), gives us

$$a^{\frac{1}{2}} \gamma_a = A^{\frac{1}{2}} \Gamma_a \qquad (a = 0,1,2) \qquad , \qquad (5.19)$$

with Γ_a the initial values of γ_a. Also, the new variables

$$\gamma^*_a \equiv a^{\frac{1}{2}} \gamma_a \qquad (a = 0,1,2) \qquad , \qquad (5.20)$$

are time constants and their material derivatives must vanish:

$$\dot{\gamma}^*_a = 0 \qquad (a = 0,1,2) \qquad . \qquad (5.21)$$

From (2.28) we have that the last result is equivalent to

$$\dot{\gamma}_a + \nabla \cdot \underset{\sim}{v} \, \gamma_a = 0 \qquad (a = 0,1,2) \qquad . \qquad (5.22)$$

We can now express the general inertia terms in the equations of motion (5.4) in terms of the two accelerations $\dot{\underset{\sim}{v}} \equiv \ddot{\underset{\sim}{x}}$ and $\dot{\underset{\sim}{\omega}} = \ddot{\underset{\sim}{n}}$; for combining the last result with the momentum relation (4.14), we obtain

$$\dot{\underset{\sim}{P}} + \nabla \cdot \underset{\sim}{v} \, \underset{\sim}{P} = \gamma_0 \, \dot{\underset{\sim}{v}} + \gamma_1 \, \dot{\underset{\sim}{\omega}} \qquad ,$$

$$(5.23)$$

$$\dot{\underset{\sim}{\ell}} + \nabla \cdot \underset{\sim}{v} \, \underset{\sim}{\ell} + \underset{\sim}{v} \times \underset{\sim}{P} = \underset{\sim}{n} \times (\gamma_1 \, \dot{\underset{\sim}{v}} + \gamma_2 \, \dot{\underset{\sim}{\omega}}) \quad .$$

With these equations, the equations of motion can be written as:

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$$\gamma_0 \; \dot{\underline{v}} + \gamma_1 \; \dot{\underline{\omega}} = \underline{Q}^{\alpha}{}_{|\alpha} + \underline{F} \quad ,$$

(5.24)

$$\underline{n} \times (\gamma_1 \; \dot{\underline{v}} + \gamma_2 \; \dot{\underline{\omega}}) = \underline{m}^{\alpha}{}_{|\alpha} + \underline{a}_{\alpha} \times \underline{Q}^{\alpha} + \underline{c} \quad .$$

The form assumed by the inertia terms in the above equations is a consequence of the Kirchhoff hypothesis. Another consequence of the Kirchhoff hypothesis is that $\underline{m}^{\alpha}$ and $\underline{c}$ must be tangential, as indicated by $(4.21)_2$ and $(4.33)_2$. Taking this result into account by replacing $\underline{m}^{\alpha}$ and $\underline{c}$ by the equivalent $\underline{M}^{\alpha}$ and $\underline{C}$, the second of the above equations becomes

$$\underline{n} \times (\gamma_1 \; \dot{\underline{v}} + \gamma_2 \; \dot{\underline{\omega}}) = \underline{n} \times \underline{a}_{\beta} \; (M^{\alpha\beta}{}_{|\alpha} + C^{\beta} - Q^{\beta})$$

(5.25)

$$+ \; \underline{a}_{\alpha} \times \underline{a}_{\beta} \; (Q^{\alpha\beta} - b^{\alpha}_{\gamma} \; M^{\gamma\beta}) \quad .$$

This equation is easily separated into components, giving for the normal component

$$0 = \varepsilon_{\alpha\beta} \; (Q^{\alpha\beta} - b^{\alpha}_{\gamma} \; M^{\gamma\beta})$$

(5.26)

and for the tangential components

$$\underline{a}^{\beta} \cdot (\gamma_1 \; \dot{\underline{v}} + \gamma_2 \; \dot{\underline{\omega}}) = M^{\alpha\beta}{}_{|\alpha} + C^{\beta} - Q^{\beta} \quad .$$

(5.27)

These equations express the balance of moment of momentum in the normal and tangential directions, respectively. From $(4.28)_{1,3}$ using (2.38), we have the identity

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$$Q^{\alpha\beta} - b_\gamma^\alpha \, M^{\gamma\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \mu_\gamma^\beta \, \mu_\delta^\alpha \, \sigma^{\gamma\delta} \, \mu \, d\zeta \qquad , \qquad (5.28)$$

which shows that if $\sigma^{\alpha\beta}$ is symmetric then (5.26) is identically satis-
fied. Hence, the tangential components of the stress tensor being
symmetric implies the balance of moment of momentum in the normal di-
rection, making (5.26) redundant. For this reason, (5.26) is often not
included among the equations of motion of Kirchhoff shells.[*] In the
next section we shall show how this equation can be utilized to affect
a symmetrization of the membrane and bending resultants.

The consequences of mass conservation can also be applied to the
equations of motion for the starred variables (5.12). Combining (4.14),
(5.11) and (5.21), we obtain

$$\dot{P}* = \gamma*_0 \, \dot{v} + \gamma*_1 \, \dot{\omega} \, , \qquad \dot{\ell}* + v \times P* = n \times (\gamma*_1 \, \dot{v} + \gamma*_2 \, \dot{\omega}) \qquad (5.29)$$

Using the last expressions to replace the inertia terms in (5.12), we
have

$$\gamma*_0 \, \dot{v} + \gamma*_1 \, \dot{\omega} = Q*^\alpha{}_{,\alpha} + F* \qquad ,$$

$$\qquad (5.30)$$

$$n \times (\gamma*_1 \, \dot{v} + \gamma*_2 \, \dot{\omega}) = m*^\alpha{}_{,\alpha} + a_\alpha \times Q*^\alpha + c* \qquad .$$

Replacing $m*^\alpha$ and $c*$ in the second of the above equations by the equi-
valent tangential vectors

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[*] See for example KRAUS [13; Eq. (2.71)].

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$$M^{*\alpha} \equiv a^{\frac{1}{2}} \underset{\sim}{M}^{\alpha} = \underset{\sim}{m}^{*\alpha} \times \underset{\sim}{n} \quad , \qquad \underset{\sim}{C}^{*} \equiv a^{\frac{1}{2}} \underset{\sim}{C} = \underset{\sim}{c}^{*} \times \underset{\sim}{n} \quad , \qquad (5.31)$$

this equation becomes

$$\underset{\sim}{n} \times (\gamma^{*}{}_{1} \underset{\sim}{\dot{v}} + \gamma^{*}{}_{2} \underset{\sim}{\dot{\omega}}) = \underset{\sim}{n} \times \underset{\sim}{a}_{\beta} (M^{*\alpha\beta}{}_{,\alpha} + \Gamma_{\alpha\gamma}{}^{\beta} M^{*\alpha\gamma} \cdot C^{*\beta} - Q^{*\beta})$$

$$\qquad (5.32)$$

$$\qquad + \underset{\sim}{a}_{\alpha} \times \underset{\sim}{a}_{\beta} (Q^{*\alpha\beta} - b^{\alpha}{}_{\gamma} M^{*\gamma\beta}) \quad ,$$

giving for its normal component

$$0 = \varepsilon_{\alpha\beta} (Q^{*\alpha\beta} - b^{\alpha}{}_{\gamma} M^{*\gamma\beta}) \qquad (5.33)$$

and for its tangential components

$$\underset{\sim}{a}^{\beta} \cdot (\gamma^{*}{}_{1} \underset{\sim}{\dot{v}} + \gamma^{*}{}_{2} \underset{\sim}{\dot{\omega}}) = M^{*\alpha\beta}{}_{,\alpha} + \Gamma_{\alpha\gamma}{}^{\beta} M^{*\alpha\gamma} + C^{*\beta} - Q^{*\beta} \quad . \qquad (5.34)$$

Equations (5.29), (5.30), (5.32)-(5.34) in the starred variables correspond to equations (5.23)-(5.27) in the unstarred variables. Again, (5.33) is implied by the symmetry of the tangential components of the stress tensor and in that sense is redundant.

The various forms of the equations of mass conservation and of the equations of motions obtained in this section are based on the premise that the shell deforms according to the Kirchhoff hypothesis. This does not mean that the resultant relations obtained in Chapter 4 for Kirchhoff shells need necessarily be used with these equations, but only that the equations are compatible with these relations. Other resultant relations can, of course, be used. However, it is clear from the special forms of the inertia terms and the tangency of the couple resultant $\underset{\sim}{m}^{\alpha}$

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or $\underline{m}^{*\alpha}$ and the external couple $\underline{c}$ or $\underline{c}^*$ to the middle surface that the equations of motion derived in this section cannot be as general as those obtained in section 5.1.

5.3 Reduced Equation of Motion

Whenever the Kirchhoff hypothesis is used, it is necessary to disregard the values given by the resultant relations $(4.28)_{1,2}$ for some components of the stress resultant Q^α. The reason for this is that with the Kirchhoff hypothesis the use of all the equations of motion and all the resultant relations yields an overdeterminate theory. To make this clear, assume that at some instant the position $\underline{r}$, the velocity field $\underline{v}$, the stress field $\underline{\sigma}$, body force field $\underline{b}$ and traction fields $\underline{t}_+$ & $\underline{t}_-$ are known. Through relations (4.21) & (4.33) the resultants $\underline{Q}^\alpha$, $\underline{m}^\alpha$, $\underline{F}$ & $\underline{c}$ will be known and consequently the right hand side of the equations of motion (4.21) completely determined. As a consequence of condition (2.31) making $\underline{\omega}$ a function of $\underline{v}_{|\alpha}$, the equations of motion become a set of five scalar partial differential equations for the three components of the acceleration $\dot{\underline{v}}$. Since the five equations are independent, this clearly implies that $\dot{\underline{v}}$ is overspecified.

It is usual to disregard the values of the shear components Q^α given by resultant relations $(4.28)_2$ and take as significant the values satisfying the tangential components of the moment of momentum equation (5.27). This is accomplished by using (5.27) to eliminate the shear components from the linear momentum equation $(5.24)_1$.[*] Since, as mentioned earlier, (5.26) is redundant and can be disregarded, with the elimination of Q^α the equations of motion are reduced to a single vector equations involving the components $Q^{\alpha\beta}$ and $M^{\alpha\beta}$.

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[*] KRAUS [13; P.47] indicates the use of this elimination without pointing out its necessity. MORINO, LEECH and WITMER [2; P.19] do mention the need for this elimination and go on to use it in developing their equations.

Rather than following this procedure to reduce the equations of motion, we shall use a more elegant method and use both (5.26) and (5.28) to eliminate the components Q^α and $Q^{[\alpha\beta]}$* from the linear momentum equation $(5.24)_1$. In doing this, we follow SANDERS [14] and LEONARD [15] by defining modified membrane components, but differ from these authors in that we employ the equations of motion rather than the equations of equilibrium and we obtain a vectorial rather than a componental representation.

We begin by defining the *modified membrane components* of the stress resultant:

$$\hat{Q}^{\alpha\beta} = Q^{\alpha\beta} + b_\gamma^\beta \, M^{\gamma\alpha} = \hat{Q}^{\beta\alpha} \quad , \tag{5.35}$$

wherein symmetry follows from (5.26). With the aid of $(2.20)_3$, we use (5.27) and (5.35) to express the stress resultant as follows:

$$\underline{Q}^\alpha = (M^{\beta\alpha} \, \underline{n})_{|\rho} + \hat{n}^{\alpha\beta} \, \underline{a}_\beta + C^\alpha \, \underline{n} - \underline{a}^\alpha \cdot (\gamma_1 \, \dot{\underline{v}} + \gamma_2 \, \dot{\underline{\omega}}) \, \underline{n}. \tag{5.36}$$

Substituting this expression into the linear momentum equation $(5.24)_1$ and using the identity

$$(M^{[\beta\alpha]} \, \underline{n})_{|\beta\alpha} \equiv 0 \quad ,^{**} \tag{5.37}$$

---

* We adapt the practice of enclosing a pair of indices in a bracket or parenthesis to denote the antisymmetric or symmetric part with respect to these indices.

** This identity easily follows from the observation that the anti-symmetric part of any two index quantity $\delta^{\alpha\beta}$ can be written as

$\delta^{[\alpha\beta]} = \epsilon^{\alpha\beta} \, \delta$, so that $\delta^{[\alpha\beta]}_{|\alpha\beta} = \epsilon^{\alpha\beta} \, \delta_{|\alpha\beta} = \epsilon^{\alpha\beta} \, (\delta_{,\alpha\beta} - \Gamma_{\alpha\beta}^\gamma \, \delta_{,\gamma}) \equiv 0$.

we obtain the *reduced equation of motion*

$$\gamma_0 \, \dot{\underline{v}} + \gamma_1 \, \dot{\underline{\omega}} + [\underline{a}^\alpha \cdot (\gamma_1 \, \dot{\underline{v}} + \gamma_2 \, \dot{\underline{\omega}}) \, \underline{n}]_{|\alpha}$$

(5.38)

$$= (\hat{M}^{\beta\alpha} \, \underline{n})_{|\beta\alpha} \, \vdots \, (\hat{Q}^{\alpha\beta} \, \underline{a}_\beta + C^\alpha \, \underline{n})_{|\alpha} + \underline{F} \quad ,$$

with $\hat{M}^{\alpha\beta}$ denoting the symmetric part of the bending components:

$$\hat{M}^{\alpha\beta} = M^{(\alpha\beta)} \equiv \frac{1}{2} \, (M^{\alpha\beta} + M^{\beta\alpha}) \quad . \tag{5.39}$$

The abbreviations

$$\hat{\underline{M}}^{\alpha\beta} = \hat{M}^{\alpha\beta} \, \underline{n} \quad , \qquad \hat{\underline{Q}}^\alpha = \hat{Q}^{\alpha\beta} \, \underline{a}_\beta \quad , \qquad \hat{\underline{C}}^\alpha = C^\alpha \, \underline{n} \quad , \tag{5.40}$$

permit us to write the reduced equation of motion more compactly:

$$\gamma_0 \, \dot{\underline{v}} + \gamma_1 \, \dot{\underline{\omega}} + [\underline{a}^\alpha \cdot (\gamma_1 \, \dot{\underline{v}} + \gamma_2 \, \dot{\underline{\omega}}) \, \underline{n}]_{|\alpha}$$

(5.41)

$$= [\hat{\underline{M}}^{\beta\alpha}{}_{|\beta} + \hat{\underline{Q}}^\alpha + \hat{\underline{C}}^\alpha]_{|\alpha} + \underline{F} \quad .$$

Setting the inertia terms on the left hand side equal to zero, we obtain
the *reduced equation of equilibrium*

$$[\hat{\underline{M}}^{\beta\alpha}{}_{|\beta} + \hat{\underline{Q}}^\alpha + \hat{\underline{C}}^\alpha]_{|\alpha} + \underline{F} = 0 \quad , \tag{5.42}$$

which is the vectorial representation of the equations derived by
SANDERS [14, Eq. 55 & 56].

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The elimination just performed reduces the equations of motion from two vector equations (5.24) (six scalar components) to a single vector equation (5.41)(three scalar components). Simultaneously, the ten variables $Q^{\alpha\beta}, Q^{\alpha}$ & $m^{\alpha\beta}$ have been replaced by six variables $\hat{Q}^{\alpha\beta}$ & $\hat{M}^{\alpha\beta}$. Had only the shear components been eliminated as indicated before, the thus reduced equation would have involved the eight variables $Q^{\alpha\beta}$ and $M^{\alpha\beta}$. We will discuss this further in section 8.1.

The resultant relations for the values of $\hat{Q}^{\alpha\beta}$ and $\hat{M}^{\alpha\beta}$ are easily determined by combining (5.35) and (5.39) with $(4.28)_{1,3}$, giving us

$$
\hat{Q}^{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} [\mu^{\alpha}_{\gamma} \sigma^{\gamma\beta} + \mu^{\beta}_{\gamma} \sigma^{\gamma\alpha} - \mu^{\alpha}_{\gamma} u^{\beta}_{\delta} \sigma^{\gamma\delta}] \mu \, d\zeta \quad ,
$$

$$(5.43)$$

$$
\hat{M}^{\alpha\beta} = \frac{1}{2} \int_{-\frac{h}{2}}^{\frac{h}{2}} [\mu^{\cdot}_{\gamma} \sigma^{\gamma\beta} + \mu^{\beta}_{\gamma} \sigma^{\gamma\alpha}] \mu \, \zeta \, d\zeta \quad .
$$

We note that the symmetry of $\hat{Q}^{\alpha\beta}$ follows from the symmetry of $\sigma^{\alpha\beta}$, which is not surprising in view of the redundancy of (5.26) due to (5.28).

We conclude by reformulating the reduced equation of motion using the starred variables:

$$
\hat{Q}*^{\alpha} = a^{\frac{1}{2}} \hat{Q}^{\alpha} \quad , \quad \hat{M}*^{\alpha\beta} = a^{\frac{1}{2}} \hat{M}^{\alpha\beta} \quad , \quad \hat{C}*^{\alpha} = a^{\frac{1}{2}} \hat{C}^{\alpha} \quad . \qquad (5.44)
$$

With the aid of (2.18) we can express (5.41) as follows

$$
\gamma*_0 \, \dot{v} + \gamma*_1 \, \dot{\omega} + [a^{\alpha} \cdot (\gamma*_1 \, \dot{v} + \gamma*_2 \, \dot{\omega}) \, n]_{,\alpha}
$$

$$(5.45)$$

$$
= [\hat{M}*^{\beta\alpha}_{,\beta} + \hat{N}*^{\alpha}]_{,\alpha} + F* \quad ,
$$

with

$$\hat{\underset{\sim}{N}}{\star}^{\alpha} = \hat{\underset{\sim}{Q}}{\star}^{\alpha} + \Gamma_{\beta\gamma}{}^{\alpha} \, \hat{\underset{\sim}{M}}{\star}^{\beta\gamma} + \hat{\underset{\sim}{C}}{\star}^{\alpha}$$

$$= \hat{Q}{\star}^{\alpha\beta} \, \underset{\sim}{a}_{\beta} + (\Gamma_{\beta\gamma}{}^{\alpha} \, \hat{M}{\star}^{\beta\gamma} + C{\star}^{\alpha}) \, \underset{\sim}{n} \quad .$$

(5.46)

6. BOUNDARY CONDITIONS FOR A SHELL

In this chapter we obtain natural boundary conditions for the re-
duced equation of motion. We do this through a variational procedure
based on the principle of virtual work. The variation in the rotation
of a shell element is assumed to satisfy the Kirchhoff hypothesis. The
three classic boundary conditions of the free, hinged and clamped edge
are obtained along general boundary curves and then specialized to the
coordinate curves. The boundary conditions along coordinate curves for
a symmetry edge are also obtained.

6.1 Variation of Strain Energy

Our goal is to find the natural boundary conditions associated with
the reduced equation of motion, using the principle of virtual work.
However, since our concern is with deriving boundary conditions rather
than initial conditions, we can disregard the inertia terms and work
just as well with the reduced equation of equilibrium (5.42). Therefore
we can state our objective more precisely as follows: to formulate a
variational problem based on the principle of virtual work that simul-
taneously yields the reduced equation of equilibrium and an associated
set of boundary conditions.

As our point of departure we use the principle of virtual work,
which for the reference surface S bounded by the simple curve C can be
expressed as follows:

$$\iint_S \delta V \, da = \iint_S (\underset{\sim}{F} \cdot \delta\underset{\sim}{r} + \underset{\sim}{c} \cdot \delta\underset{\sim}{\theta}) \, da + \oint_C (\underset{\sim}{Q} \cdot \delta\underset{\sim}{r} + \underset{\sim}{m} \cdot \delta\underset{\sim}{\theta}) \, ds , \qquad (6.1)$$

with δV the variation in strain energy per unit surface area, $\delta\underset{\sim}{r}$ the
variation in the position vector and $\delta\underset{\sim}{\theta}$ the rotational part of the
variation in the basis $(\underset{\sim}{a}_\alpha, \underset{\sim}{n})$. $\underset{\sim}{Q}$ and $\underset{\sim}{m}$ are the *contact force* and

contact couple per unit arc length applied along the boundary C and, as already mentioned, F and c are the external force and external couple per unit area over the surface S.

The rotational part $\delta\theta$ of the variation in the basis (a_α, n) is given by the expression

$$\delta\theta = \frac{1}{2} (a^\alpha \times \delta a_\alpha + n \times \delta n) \quad .^* \tag{6.2}$$

This expression assumes no dependence of δn on δa_α, even though n is normal to a_α. We set up such a dependence by requiring that the variation in (a_α, n) satisfy the Kirchhoff hypothesis. Consequently the variations δn and δa_α must be such that the magnitude of n is unchanged

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*This expression is easily derived by considering any vector β with fixed components relative to the basis (a_α, n):

$$\beta = \beta^\alpha a_\alpha + \beta n \equiv \beta \cdot a^\alpha a_\alpha + \beta \cdot n n \quad .$$

Its variation due to varying (a_α, n) is

$$\delta\beta = \beta^\alpha \delta a_\alpha + \beta \delta n \equiv \beta \cdot a^\alpha \delta a_\alpha + \beta \cdot n \delta n$$

and the rotational part of $\delta\beta$ is given by the antisymmetric part:

$$\delta\theta \times \beta = \frac{1}{2} \beta \cdot (a^\alpha \delta a_\alpha - \delta a_\alpha a^\alpha + n \delta n - \delta n n) \quad ,$$

from which the above expression follows.

and $\underset{\sim}{n}$ remains normal to $\underset{\sim}{a}$ during the variation, which gives us the conditions

$$\underset{\sim}{n} \cdot \delta\underset{\sim}{n} = 0 \quad , \quad \underset{\sim}{a}_\alpha \cdot \delta\underset{\sim}{n} + \underset{\sim}{n} \cdot \delta\underset{\sim}{a}_\alpha = 0 \quad . \tag{6.3}$$

Combining these conditions with the expression for $\delta\theta$, we have as a consequence of the Kirchhoff hypothesis that $\delta\theta$ is completely determined by $\delta\underset{\sim}{a}_\alpha$:

$$\delta\underset{\sim}{\theta} = \frac{1}{2} \underset{\sim}{a}^\alpha \times (\delta\underset{\sim}{a}_\alpha + \underset{\sim}{n}\,\underset{\sim}{n} \cdot \delta\underset{\sim}{a}_\alpha) \quad . \tag{6.4}$$

Our aim is to find an appropriate expression for δV; one which when substituted in the principle of virtual work (6.1) yields the reduced equation of equilibrium. We begin with the requirement that the principle of virtual work be consistent with the vanishing of the expression

$$\delta A = (\underset{\sim}{Q}^\alpha{}_{|\alpha} + \underset{\sim}{F}) \cdot \delta\underset{\sim}{r} + (\underset{\sim}{m}^\alpha{}_{|\alpha} + \underset{\sim}{a}_\alpha \times \underset{\sim}{Q}^\alpha + \underset{\sim}{c}) \cdot \delta\underset{\sim}{\theta} \quad . \tag{6.5}$$

Motivation for this requirement follows from the fact that δA does indeed vanish when the equations of equilibrium closest to physical reality (5.5) are satisfied. Our analysis, however, does not depend upon (5.5) being satisfied, but only on (6.5) vanishing.

A change in the order of differentiation in the last expression gives

$$\delta A = (\underset{\sim}{Q}^\alpha \cdot \delta\underset{\sim}{r} + \underset{\sim}{m}^\alpha \cdot \delta\underset{\sim}{\theta})_{|\alpha} + \underset{\sim}{F} \cdot \delta\underset{\sim}{r} + \underset{\sim}{c} \cdot \delta\underset{\sim}{\theta}$$

$$\tag{6.6}$$

$$- (\underset{\sim}{Q}^\alpha \cdot \delta\underset{\sim}{a}_\alpha - \underset{\sim}{a}_\alpha \times \underset{\sim}{Q}^\alpha \cdot \delta\underset{\sim}{\theta} + \underset{\sim}{m}^\alpha \cdot \delta\underset{\sim}{\theta}_{|\alpha}) \quad .$$

With the aid of (2.16) and (2.20), the derivative of $\delta\underset{\sim}{\theta}$ can be written as follows

$$\delta\underset{\sim}{\theta}_{|\alpha} = \frac{1}{2} [\underset{\sim}{n} \times \underset{\sim}{a}^\beta (b^\gamma_\alpha \, \delta a_{\gamma\beta} - 2 \, \delta b_{\alpha\beta}) + \underset{\sim}{a}^\beta \times \underset{\sim}{a}_\gamma \, \delta\Gamma_{\alpha\beta}{}^\gamma] \quad , \qquad (6.7)$$

which when combined with (4.24) gives us

$$\underset{\sim}{m}^\alpha \cdot \delta\underset{\sim}{\theta}_{|\alpha} = \frac{1}{2} b^\alpha_\gamma M^{\gamma\beta} \, \delta a_{\alpha\beta} - M^{\alpha\beta} \, \delta b_{\alpha\beta} \quad . \qquad (6.8)$$

We also have from (6.4) the identity

$$\underset{\sim}{Q}^\alpha \cdot \delta\underset{\sim}{a}_\alpha - \underset{\sim}{a}_\alpha \times \underset{\sim}{Q}^\alpha \cdot \delta\underset{\sim}{\theta} = \frac{1}{2} Q^{\alpha\beta} \, \delta a_{\alpha\beta} \quad . \qquad (6.9)$$

In view of definitions (5.35) and (5.39), substitution of the last two expressions in (6.6) results in

$$\delta A = (\underset{\sim}{Q}^\alpha \cdot \delta\underset{\sim}{r} + \underset{\sim}{m}^\alpha \cdot \delta\underset{\sim}{\theta})_{|\alpha} + \underset{\sim}{F} \cdot \delta\underset{\sim}{r} + \underset{\sim}{c} \cdot \delta\underset{\sim}{\theta}$$

$$\qquad\qquad\qquad\qquad\qquad (6.10)$$

$$- (\frac{1}{2} \hat{Q}^{\alpha\beta} \, \delta a_{\alpha\beta} - \hat{M}^{\alpha\beta} \, \delta b_{\alpha\beta}) \quad .$$

Integrating this equation over the surface S and applying the divergence theorem for surfaces,[*] we obtain

$$\iint\limits_S \delta A \, da = \oint\limits_C (\underset{\sim}{Q}^\alpha \cdot \delta\underset{\sim}{r} + \underset{\sim}{m}^\alpha \cdot \delta\underset{\sim}{\theta}) \, \nu_\alpha \, ds + \iint\limits_S (\underset{\sim}{F} \cdot \delta\underset{\sim}{r} + \underset{\sim}{c} \cdot \delta\underset{\sim}{\theta}) \, da$$

$$\qquad\qquad\qquad\qquad\qquad (6.11)$$

$$- \iint\limits_S (\frac{1}{2} \hat{Q}^{\alpha\beta} \, \delta a_{\alpha\beta} - \hat{M}^{\alpha\beta} \, \delta b_{\alpha\beta}) \, da \quad .$$

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[*]See Appendix A for a derivation of this theorem.

Comparison of this result with (6.1) shows that with

$$\delta V = \frac{1}{2} \hat{Q}^{\alpha\beta} \, \delta a_{\alpha\beta} - \hat{M}^{\alpha\beta} \, \delta b_{\alpha\beta} \quad , \tag{6.12}$$

the principle of virtual work and the vanishing of δA are consistent.

We have yet to show that this choice of δV implies the reduced equation of equilibrium (5.42), although the appearance in (6.12) of precisely those mechanical variables occurring in the reduced equation strongly suggests such a connection. Returning to (6.5), we use (4.24) and (4.34) to replace $\underset{\sim}{m}^{\alpha}$ and $\underset{\sim}{c}$ by $\underset{\sim}{M}^{\alpha}$ and $\underset{\sim}{C}$, giving us

$$\delta A = [Q^{\alpha}_{|\alpha} + \underset{\sim}{F}] \cdot \delta\underset{\sim}{r} + [\underset{\sim}{n} \times (M^{\alpha}_{|\alpha} + \underset{\sim}{C}) + \underset{\sim}{a}_{\alpha} \times (\underset{\sim}{Q}^{\alpha} - b^{\alpha}_{\beta} \underset{\sim}{M}^{\beta})] \cdot \delta\underset{\sim}{\theta} \; . \tag{6.13}$$

Using (6.4) to eliminate $\delta\underset{\sim}{\theta}$ and rearranging, this becomes

$$\delta A = [Q^{\alpha}_{|\alpha} + \underset{\sim}{F}] \cdot \delta\underset{\sim}{r} - [(M^{\beta\alpha} \underset{\sim}{n})_{|\beta} + (Q^{(\alpha\beta)} + b^{(\alpha}_{\gamma} M^{\gamma\beta)}) \underset{\sim}{a}_{\beta}. \tag{6.14}$$

$$+ \, \underset{\sim}{C}^{\alpha} \, \underset{\sim}{n} - \underset{\sim}{Q}^{\alpha}] \cdot \delta\underset{\sim}{a}_{\alpha} \quad .$$

Using the fact that $\delta\underset{\sim}{a}_{\alpha} \equiv \delta\underset{\sim}{r}_{|\alpha}$ to change the order of differentiation, we obtain, in view of definitions (5.35), (5.39) and (5.40), the result

$$\delta A = [(\hat{M}^{\beta\alpha}_{|\beta} + \hat{Q}^{\alpha} + \hat{C}^{\alpha})_{|\alpha} + \underset{\sim}{F}] \cdot \delta\underset{\sim}{r} \tag{6.15}$$

$$- \, \{[(M^{\beta\alpha} \underset{\sim}{n})_{|\beta} + \hat{Q}^{\alpha} + \hat{C}^{\alpha} - \underset{\sim}{Q}^{\alpha}] \cdot \delta\underset{\sim}{r}\}_{|\alpha} \quad .$$

Combining (4.24) and (6.4) to obtain

$$\underset{\sim}{m}^{\alpha} \cdot \delta\underset{\sim}{\theta} = - M^{\alpha\beta} \underset{\sim}{n} \cdot \delta\underset{\sim}{a}_{\beta} \tag{6.16}$$

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and substituting this result and the previous expression for δA in (6.11), we obtain after rearranging terms the integral identity

$$\iint\limits_{S} (\tfrac{1}{2} \hat{Q}^{\alpha\beta} \, \delta a_{\alpha\beta} - \hat{M}^{\alpha\beta} \, \delta b_{\alpha\beta}) \, da + \iint\limits_{S} [(\hat{M}^{\beta\alpha}{}_{|\beta} + \hat{Q}^{\alpha} + \hat{C}^{\alpha})_{|\alpha} + \underline{F}] \cdot \delta\underline{r} \, da$$

$$= \iint\limits_{S} (\underline{F} \cdot \delta\underline{r} + \underline{c} \cdot \delta\underline{\theta}) \, da + \oint\limits_{C} \{[(M^{\beta\alpha} \, \underline{n})_{|\beta} + \hat{Q}^{\alpha} + \hat{C}^{\alpha}] \cdot \delta\underline{r} \tag{6.17}$$

$$- M^{\alpha\beta} \, \underline{n} \cdot \delta\underline{a}_{\beta}\} \, \nu_{\alpha} \, ds \quad ,$$

which on identifying δV as per (6.12) becomes

$$\iint\limits_{S} \delta V \, da + \iint\limits_{S} [(\hat{M}^{\beta\alpha}{}_{|\beta} + \hat{Q}^{\alpha} + \hat{C}^{\alpha})_{|\alpha} + \underline{F}] \cdot \delta\underline{r} \, da$$

$$= \iint\limits_{S} (\underline{F} \cdot \delta\underline{r} + \underline{c} \cdot \delta\underline{\theta}) \, da + \oint\limits_{C} \{[(M^{\beta\alpha} \, \underline{n})_{|\beta} + \hat{Q}^{\alpha} + \hat{C}^{\alpha}] \cdot \delta\underline{r} \tag{6.18}$$

$$- M^{\alpha\beta} \, \underline{n} \cdot \delta\underline{a}_{\beta}\} \, \nu_{\alpha} \, ds \quad .$$

From comparing this result with (6.1), we conclude that the principle of virtual work implies the integral equation

$$\iint\limits_{S} [(\hat{M}^{\beta\alpha}{}_{|\beta} + \hat{Q}^{\alpha} + \hat{C}^{\alpha})_{|\alpha} + \underline{F}] \cdot \delta\underline{r} \, da + \oint\limits_{C} (\underline{Q} \cdot \delta\underline{r} + \underline{m} \cdot \delta\underline{\theta}) \, ds$$

$$\tag{6.19}$$

$$= \oint\limits_{C} \{[(M^{\beta\alpha} \, \underline{n})_{|\beta} + \hat{Q}^{\alpha} + \hat{C}^{\alpha}] \cdot \delta\underline{r} - M^{\alpha\beta} \, \underline{n} \cdot \delta\underline{a}_{\beta}\} \, \nu_{\alpha} \, ds.$$

Since $\delta\underline{r}$ is arbitrary, we first set $\delta\underline{r} = 0$ over S, giving us

$$\oint\limits_C (\underline{Q} \cdot \delta\underline{r} + \underline{m} \cdot \delta\underline{\theta})\, ds = \oint\limits_C \{[(M^{\beta\alpha}\, \underline{n})_{|\beta} + \hat{\underline{Q}}^\alpha + \hat{\underline{C}}^\alpha] \cdot \delta\underline{r}$$

$$-\, M^{\alpha\beta}\, \underline{n} \cdot \delta\underline{a}_\beta\}\, \nu_\alpha\, ds \quad .$$

(6.20)

It then follows from (6.19) that for arbitrary $\delta\underline{r}$

$$\iint\limits_S [(\hat{\underline{M}}^{\beta\alpha}_{\ |\beta} + \hat{\underline{Q}}^\alpha + \hat{\underline{C}}^\alpha)_{|\alpha} + \underline{F}] \cdot \delta\underline{r}\, da = 0 \qquad (6.21)$$

Assuming that the integrand of this expression is continuous, the arbitrariness of $\delta\underline{r}$ implies that the integrand itself must vanish, which is clearly equivalent to the reduced equation of equilibrium (5.42) being satisfied. Hence we have shown that with the choice of (6.12) for δV the principle of virtual work does indeed imply the reduced equation of equilibrium, and so (6.20) contains the associated natural boundary conditions.

6.2 Classical Boundary Conditions alon̲ ̲̲eral Curves

As it stands, expression (6.20) is somewhat unsatisfactory in that, while the reduced equation of equilibrium (5.42) and the variation in strain energy (6.12) depend only on the symmetric part of $M^{\alpha\beta}$, it appears that (6.20) involves $M^{\alpha\beta}$ entirely. We now show that this dependence is spurious by first writing, without loss of generality, the antisymmetric part of $M^{\alpha\beta}$ as

$$M^{[\alpha\beta]} = \varepsilon^{\alpha\beta}\, M \qquad . \qquad (6.22)$$

It then follows that

$$\oint_C [(M^{[\alpha\beta]} \, \underset{\sim}{n})_{|\beta} \cdot \delta\underset{\sim}{r} - M^{[\beta\alpha]} \, \underset{\sim}{n} \cdot \delta\underset{\sim}{a}_\beta] \, \nu_\alpha \, ds$$

$$= \oint_C (M \, \underset{\sim}{n} \cdot \delta\underset{\sim}{r})_{,\beta} \, \epsilon^{\beta\alpha} \, \nu_\alpha \, ds \quad , \tag{6.23}$$

which by (A.9) becomes

$$\oint_C [(M^{[\alpha\beta]} \, \underset{\sim}{n})_{|\beta} \cdot \delta\underset{\sim}{r} - M^{[\beta\alpha]} \, \underset{\sim}{n} \cdot \delta\underset{\sim}{a}_\beta] \, \nu_\alpha \, ds$$

$$= - \oint_C (M \, \underset{\sim}{n} \cdot \delta\underset{\sim}{r})_{,\beta} \, \tau^\beta \, ds \quad . \tag{6.24}$$

Since $\underset{\sim}{\tau}$ is the unit tangent to C and satisfies

$$\underset{\sim}{\tau} = \frac{d\underset{\sim}{r}}{ds} \equiv \frac{d\xi^\alpha}{ds} \, \underset{\sim}{a}_\alpha \quad , \tag{6.25}$$

we have as a consequence of $M \, \underset{\sim}{n} \cdot \delta\underset{\sim}{r}$ being continuous that

$$\oint_C [(M^{[\beta\alpha]} \, \underset{\sim}{n})_{|\beta} \cdot \delta\underset{\sim}{r} - M^{[\alpha\beta]} \, \underset{\sim}{n} \cdot \delta\underset{\sim}{a}_\beta] \, \nu_\alpha \, ds$$

$$= - \oint_C \frac{d}{ds} (M \, \underset{\sim}{n} \cdot \delta\underset{\sim}{r}) \, ds \equiv 0 \quad , \tag{6.26}$$

and hence that (6.20) depends only on the symmetric part of $M^{\alpha\beta}$.

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Consequently, by virtue of $(5.40)_1$ we can write

$$\oint_C (Q \cdot \delta r + m \cdot \delta\theta)\, ds = \oint_C [(\hat{M}^{\beta\alpha}{}_{|\beta} + \hat{Q}^\alpha + \hat{C}^\alpha) \cdot \delta r - \hat{M}^{\beta\alpha} \cdot \delta a_\beta]\, \nu_\alpha\, ds\ .$$

(6.27)

Because m^α has no normal component, see $(4.21)_2$, we can assume without loss of generality that the contact couple m also lacks a normal component

$$m \cdot n = 0\ .$$

(6.28)

Hence, we can define the equivalent vector M:

$$M = m \times n \quad , \qquad m = n \times M \quad ,$$

(6.29)

from which it follows that

$$m \cdot \delta\theta = - M \cdot a^\beta\, n \cdot \delta a_\beta = - M^\beta\, n \cdot \delta a_\beta \quad .$$

(6.30)

Substituting this expression in (6.27), we obtain

$$\oint_C \{[(\hat{M}^{\beta\alpha}{}_{|\beta} + \hat{Q}^\alpha + \hat{C}^\alpha)\nu_\alpha - Q] \cdot \delta r - [\hat{M}^{\beta\alpha}\nu_\alpha - M^\beta]\, n \cdot \delta a_\beta\}\, ds = 0\ .$$

(6.31)

The variation in a_β in the above equation cannot be independent of δr because $a_\alpha \equiv r,_\alpha$. To isolate this dependence, we first observe that by the Kirchhoff relation $(6.3)_1$ δn cannot have a normal component and hence can be resolved tangential to S along and normal to the curve C:

$$\delta n = \tau\, \tau \cdot \delta n + \nu\, \nu \cdot \delta n \quad .$$

(6.32)

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The component $\underset{\sim}{\nu} \cdot \delta\underset{\sim}{n}$ represents a variational rotation of the normal $\underset{\sim}{n}$ about the tangent vector $\underset{\sim}{\tau}$ and is independent of $\delta\underset{\sim}{r}$ in the sense that $\underset{\sim}{\nu} \cdot \delta\underset{\sim}{n}$ can vary arbitrarily when $\delta\underset{\sim}{r} = 0$. Hence we write this component as a rotation:

$$\delta\phi = \underset{\sim}{\nu} \cdot \delta\underset{\sim}{n} \qquad . \qquad (6.33)$$

The component $\underset{\sim}{\tau} \cdot \delta\underset{\sim}{n}$ however is entirely determined by $\delta\underset{\sim}{r}$, since

$$\underset{\sim}{\tau} \cdot \delta\underset{\sim}{n} = - \underset{\sim}{n} \cdot \delta\underset{\sim}{\tau} = - \underset{\sim}{n} \cdot \delta\frac{d\underset{\sim}{r}}{ds} = - \underset{\sim}{n} \cdot \frac{d\delta\underset{\sim}{r}}{ds} \qquad . \qquad (6.34)$$

With the last two expressions $\delta\underset{\sim}{n}$ becomes

$$\delta\underset{\sim}{n} = \delta\phi \; \underset{\sim}{\nu} - \underset{\sim}{n} \cdot \frac{d\delta\underset{\sim}{r}}{ds} \underset{\sim}{\tau} \qquad , \qquad (6.35)$$

and so by the Kirchhoff relation $(6.3)_2$ we have

$$\underset{\sim}{n} \cdot \delta\underset{\sim}{a}_\beta = \tau_\beta \underset{\sim}{n} \cdot \frac{d\delta\underset{\sim}{r}}{ds} - \nu_\beta \; \delta\phi \qquad . \qquad (6.36)$$

Substituting this result into the boundary integral (6.31) and integrating by parts along C, we obtain

$$\oint_C \{[(\hat{M}^{\beta\alpha}{}_{|\beta} + \hat{Q}^\alpha + \hat{C}^\alpha) \; \nu_\alpha - \underset{\sim}{Q} + \frac{d}{ds} (\hat{M}^{\beta\alpha} \; \nu_\alpha \; \tau_\beta - M^\beta \; \underset{\sim}{n} \; \tau_\beta)] \cdot \delta\underset{\sim}{r}$$

$$(6.37)$$

$$+ [(\hat{M}^{\beta\alpha} \; \nu_\alpha - M^\beta) \; \nu_\beta] \; \delta\phi\} \; ds - \sum_i [(\hat{M}^{\beta\alpha} \; \nu_\alpha - M^\beta \; \underset{\sim}{n}) \; \tau_\beta] \cdot \delta\underset{\sim}{r} \; \Big|_{s_i^+}^{\bar{s}_{i+1}} = 0,$$

wherein the last sum is over points i of discontinuity in the tangent $\underset{\sim}{\tau}$, and hence in the normal $\underset{\sim}{\nu}$, and possibly in the applied contact couple

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component M^β. Using the arbitrariness of δr, we can separate the last equation; first setting $\delta r = 0$, we get

$$\oint_C [(\hat{M}^{\beta\alpha} \nu_\alpha - M^\beta) \nu_\beta] \, \delta\phi \, ds = 0 \qquad , \tag{6.38}$$

which when substituted back in (6.37) gives us for arbitrary δr

$$\oint_C \{[(\hat{M}^{\beta\alpha}{}_{|\beta} + \hat{Q}^\alpha + \hat{C}^\alpha) \nu_\alpha - Q] + \frac{d}{ds} [(\hat{M}^{\beta\alpha} \nu_\alpha - M^\beta \, n) \, \tau_\beta]\} \cdot \delta r \, ds \tag{6.39}$$

$$+ \sum_i [\tau_\beta^{(+)} (\hat{M}^{\beta\alpha} \nu_\alpha^{(+)} - M^{(+)\beta}) - \tau_\beta^{(-)} (\hat{M}^{\beta\alpha} \nu_\alpha^{(-)} - M^{(-)\beta})] \, n \cdot \delta r = C,$$

wherein we use superscripts $(+)$ and $(-)$ to denote the positive and negative one-sided limits.[*]

Using the boundary conditions (6.38) and (6.39) we obtain the classic boundary conditions along a general curve C:

1. *Clamped or cantileverred edge*, characterized by the position r and the normal n being fixed along the boundary; hence along any portion of the boundary that is clamped (6.38) and (6.39) are automatically satisfied, since $\delta r \equiv 0$ and $\delta\phi \equiv 0$, and we have along a clamped edge

$$r = \text{const.} \qquad , \qquad n = \text{const.} \tag{6.40}$$

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[*] $\oint^{(+)} = \lim_{s \downarrow s_i} \oint(s)$ and $\oint^{(-)} = \lim_{s \uparrow s_i} \oint(s)$, cf. BUCK [16 ; p.79].

2. *Hinged or simply supported edge*, characterized by the contact couple
    M vanishing and the position r being fixed along the boundary; hence
    along any hinged portion of the boundary (6.39) is satisfied, since
    $\delta r \equiv 0$. Assuming the integrand continuous, (6.38) will be satisfied
    for arbitrary $\delta\phi$ only if the integrand vanishes, so that along a
    hinged edge

$$r = \text{const.} \quad , \quad \hat{M}^{\alpha\beta} \nu_\alpha \nu_\beta = 0 \quad . \qquad (6.41)$$

3. *Free edge*, characterized by the contact force Q and contact couple
    M vanishing along the boundary, edge displacements and rotations
    being arbitrary. Hence, assuming the integrands continuous, (6.38)
    and (6.39) will be satisfied along *smooth* portions of the boundary
    only if

$$(\hat{M}^{\beta\alpha}{}_{|\beta} + \hat{Q}^\alpha + \hat{C}^\alpha)\,\nu_\alpha + \frac{d}{ds}\,(\hat{M}^{\beta\alpha}\,\nu_\alpha\,\tau_\beta) = 0, \quad \hat{M}^{\alpha\beta}\,\nu_\alpha\,\nu_\beta = 0 \quad , \quad (6.42)$$

and at corners where τ is discontinuous only if

$$\hat{M}^{\beta\alpha}\,(\nu_\alpha^{(+)}\,\tau_\beta^{(+)} - \nu_\alpha^{(-)}\,\tau_\beta^{(-)}) = 0 \quad . \qquad (6.43)$$

6.3 Boundary Conditions along Coordinate Curves

Along coordinate curves, where either ξ^1 or ξ^2 is a constant, the
free edge conditions and to a smaller extent the hinged edge conditions
take on simpler forms. Let us first consider a $\xi^1 = $ const. coordinate
curve as a boundary. Along such a curve, the curve tangent τ is colinear
with a_2 and the curve normal ν is colinear with a^1. Since τ and ν are
unit vectors, we have that

$$\tau^1 = 0 \, , \quad \tau^2 = \pm 1/\sqrt{a_{22}} \, , \quad \nu_1 = \pm 1/\sqrt{a^{11}} \, , \quad \nu_2 = 0 \, , \qquad (6.44)$$

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with the upper sign for the case when $\underset{\sim}{\nu}$ and $\underset{\sim}{a}^1$ have the same sense and the lower when opposite. Moreover,

$$\tau_1 = a_{12} \tau^2 \quad , \quad \tau_2 = a_{22} \tau^2 \quad , \quad \frac{d}{ds} = \tau^2 \frac{\partial}{\partial \xi^2} \quad . \tag{6.45}$$

Substituting these results in (6.41) and (6.42), we obtain for edge conditions along $\xi^1 = $ const:

2'. *Hinged edge* $(\xi^1 = $ const)

$$\underset{\sim}{r} = \text{const.} \quad , \quad \hat{M}*^{11} = 0 \quad , \tag{6.46}$$

3'. *Free edge* $(\xi^1 = $ const)

$$\hat{M}*^{11}{}_{,1} + 2 \hat{M}*^{12}{}_{,2} + \hat{N}*^1 = 0 \quad , \quad \hat{M}*^{11} = 0 \quad , \tag{6.47}$$

wherein the starred variables defined by (5.44) and (5.46) are most conveniently used. It is interesting to observe after writing the reduced equation of motion (5.45) in extended form

$$\gamma*_0 \underset{\sim}{\dot{v}} + \gamma*_1 \underset{\sim}{\dot{\omega}} + [\underset{\sim}{a}^\alpha \cdot (\gamma*_1 \underset{\sim}{\dot{v}} + \gamma*_2 \underset{\sim}{\dot{\omega}}) \, \underset{\sim}{n}]_{,\alpha}$$

$$\tag{6.48}$$

$$= (\hat{M}*^{11}{}_{,1} + 2 \hat{M}*^{12}{}_{,2} + \hat{N}*^1)_{,1} + (\hat{M}*^{22}{}_{,2} + \hat{N}*^2)_{,2} + \underset{\sim}{F}* \quad ,$$

that precisely those variables differentiated with respect to ξ^1 in the reduced equation are the ones that vanish along the free edge.

Next taking a $\xi^2 = $ const. coordinate curve as a boundary, we have

$$\tau^1 = \mp 1/\sqrt{a_{11}} \ , \qquad \tau^2 = 0 \ , \qquad \nu_1 = 0 \ , \qquad \nu_2 = \pm 1/\sqrt{a_{22}} \ , \qquad (6.49)$$

with the upper sign when $\underset{\sim}{\nu}$ and $\underset{\sim}{a}^2$ have the same sense and the lower sign otherwise. Moreover,

$$\tau_1 = a_{11} \tau^1 \quad , \quad \tau_2 = a_{21} \tau^1 \quad , \quad \frac{d}{ds} = \tau^1 \frac{\partial}{\partial \xi^1} \ . \qquad (6.50)$$

Substituting these results in (6.41) and (6.42), we obtain

2". *Hinged edge* ($\xi^2 = $ const)

$$\underset{\sim}{r} = \text{const.} \quad , \qquad \hat{M}^{\star 22} = 0 \quad . \qquad (6.51)$$

3". *Free edge* ($\xi^2 = $ const)

$$\hat{M}^{\star 22}{}_{,2} + 2 \hat{M}^{\star 12}{}_{,1} + \hat{N}^{\star 2} = 0 \quad , \qquad \hat{M}^{\star 22} = 0 \quad . \qquad (6.52)$$

Rearranging the reduced equation of motion (6.48):

$$\gamma^{\star}{}_0 \underset{\sim}{\dot{v}} + \gamma^{\star}{}_1 \underset{\sim}{\dot{\omega}} + [\underset{\sim}{a}^{\alpha} \cdot (\gamma^{\star}{}_1 \underset{\sim}{\dot{v}} + \gamma^{\star}{}_2 \underset{\sim}{\dot{\omega}}) \underset{\sim}{n}]_{,\alpha}$$

$$\qquad (6.53)$$

$$= (\hat{M}^{\star 22}{}_{,2} + 2 \hat{M}^{\star 12}{}_{,1} + \hat{N}^{\star 2})_{,2} + (\hat{M}^{\star 11}{}_{,1} + \hat{N}^{\star 1})_{,1} + \underset{\sim}{F}^{\star} \ ,$$

we again observe that the variables differentiated with respect to ξ^2 vanish along the free edge.

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When a boundary coincides with the intersection of a ξ^1 = const. and a ξ^2 = const.coordinate curve, then the free edge condition at corners (6.43) holds, giving us in addition to conditions (6.47) and (6.52) the condition

3'". *Free corner* (ξ^1 = const & ξ^2 = const.)

$$\hat{M}*^{12} = 0 \quad .$$ (6.54)

6.4 Symmetry Boundary Conditions

A *symmetry edge or boundary* is not a physical boundary of the shell, but is a middle surface curve that always lies in a plane about which the shell deforms symmetrically. Hence, initially and for all subsequent positions, this plane will be a symmetry plane of the shell. It is convenient to take the symmetry boundary as a coordinate curve and to have the remaining coordinate curves symmetrically distributed with respect to the symmetry plane. Also the external force $\underset{\sim}{F}$ and external couple $\underset{\sim}{c}$ must be symmetrically distributed.

Let us first take the symmetry edge to be the $\xi^1 = \xi^1_0$ coordinate curve. Also let the symmetry plane be located a perpendicular distance K_1 from the origin, and oriented relative to the constant orthonormal basis $\underset{\sim}{k}_i$ (i = 1,2,3) such that $\underset{\sim}{k}_1$ is normal and hence $\underset{\sim}{k}_\sigma$ (σ = 2,3) parallel to the plane as shown in Figure 5.1. It then follows that when $\xi^1 = \xi^{\cdot}_0$, the position vector lies on the symmetry plane, so that

$$\underset{\sim}{r} \; (\xi^1_0, \; \xi^2, \; t) \; \cdot \; \underset{\sim}{k}_1 = K_1$$ (6.55)

Moreover, using the simplifying notation

$$\oint_+ = \oint (\xi^1_0 + \Delta\xi^1, \; \xi^2), \quad \oint_- = \oint (\xi^1_0 - \Delta\xi^1, \; \xi^2), \quad \hat{n} = \hat{n} \; (\xi^1_0, \; \xi^2) \; ,$$

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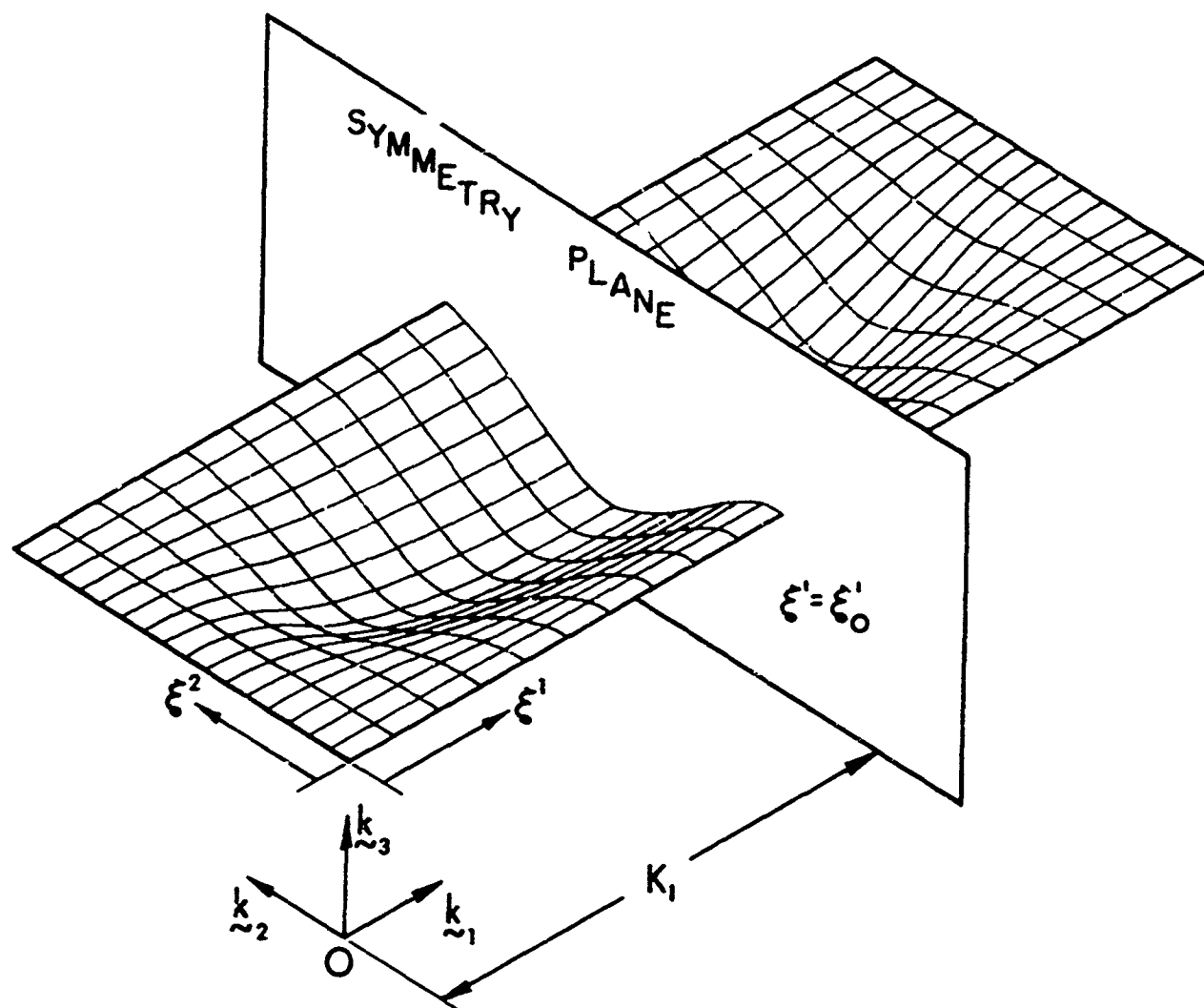
Figure 6.1 Deflection pattern of the middle surface of a shell deforming symmetrically about a symmetry plane. The intersection of the symmetry plane and the middle surface is the symmetry boundary, for which material coordinate is a constant $\xi^1 = \xi_0^1$. The symmetry plane is a perpendicular distance K_1 from the origin O and normal to the vector k_1 of the constant orthonormal basis k_i.

we have, as a consequence of the coordinate curves being symmetrically located with respect to the symmetry plane, that

$$(\underset{\sim}{r}_+ + \underset{\sim}{r}_-) \cdot \underset{\sim}{k}_1 = 2 \underset{\sim}{r} \cdot \underset{\sim}{k}_1 = 2 K_1 \ , \quad (\underset{\sim}{r}_+ - \underset{\sim}{r}_-) \cdot \underset{\sim}{k}_\sigma = 0 \quad (\sigma = 2,3).$$

$$(6.56)$$

With these relations, we obtain in Appendix B the symmetry properties of the basis vectors, normal vector, metric, etc. and combine these results with the velocity relations, strain relations and resultant relations to obtain the symmetry properties of the variables appearing in the reduced equation of motion (5.45):

$$(\underset{\sim}{a}_+^1 - \underset{\sim}{a}_-^1) \cdot \underset{\sim}{k}_1 = 0 \quad , \qquad (\underset{\sim}{a}_+^1 + \underset{\sim}{a}_-^1) \cdot \underset{\sim}{k}_\sigma = 0 \quad ,$$

$$(\underset{\sim}{a}_+^2 + \underset{\sim}{a}_-^2) \cdot \underset{\sim}{k}_1 = 0 \quad , \qquad (\underset{\sim}{a}_+^2 - \underset{\sim}{a}_-^2) \cdot \underset{\sim}{k}_\sigma = 0 \quad , \qquad (6.57)$$

$$(\underset{\sim}{n}_+ + \underset{\sim}{n}_-) \cdot \underset{\sim}{k}_1 = 0 \quad , \qquad (\underset{\sim}{n}_+ - \underset{\sim}{n}_-) \cdot \underset{\sim}{k}_\sigma = 0 \quad ,$$

$$(\underset{\sim}{v}_+ + \underset{\sim}{v}_-) \cdot \underset{\sim}{k}_1 = 0 \quad , \qquad (\underset{\sim}{v}_+ - \underset{\sim}{v}_-) \cdot \underset{\sim}{k}_\sigma = 0 \quad ,$$

$$(\underset{\sim}{\omega}_+ + \underset{\sim}{\omega}_-) \cdot \underset{\sim}{k}_1 = 0 \quad , \qquad (\underset{\sim}{\omega}_+ - \underset{\sim}{\omega}_-) \cdot \underset{\sim}{k}_\sigma = 0 \quad , \qquad (6.58)$$

$$(\underset{\sim}{F}_+^* + \underset{\sim}{F}_-^*) \cdot \underset{\sim}{k}_1 = 0 \quad , \qquad (\underset{\sim}{F}_+^* - \underset{\sim}{F}_-^*) \cdot \underset{\sim}{k}_\sigma = 0 \quad ,$$

$$\hat{M}_+^{*11} = \hat{M}_-^{*11} \quad , \quad \hat{M}_+^{*12} = - \hat{M}_-^{*12} \quad , \quad \hat{M}_+^{*22} = \hat{M}_-^{*22} \quad ,$$

$$\hat{Q}_+^{*11} = \hat{Q}_-^{*11} \quad , \quad \hat{Q}_+^{*12} = - \hat{Q}_-^{*12} \quad , \quad \hat{Q}_+^{*22} = \hat{Q}_-^{*22} \quad , \qquad (6.59)$$

$$\hat{N}_+^{*1} = - \hat{N}_-^{*1} \quad , \qquad \hat{N}_+^{*2} = \hat{N}_-^{*2} \quad .$$

Clearly, the above symmetry relations hold at positions symmetrically located with respect to the boundary and are useful in the finite differrence formulation of the reduced equations of motion along symmetry boundaries. These relations can be also be used to obtain the conditions on these variables and their derivatives at the boundary, for observe that as $\Delta\xi^1 \to 0$, we have that $\delta_+ \to \delta$ and $\delta_- \to \delta$. When we use this fact and note that $\underline{a}_1$ and $\underline{a}^1$ at the boundary are colinear with $\underline{k}_1$ we find that

$$v^1 = 0 \; , \; \omega^1 = 0 \; , \; F^{*1} = 0 \; , \; \hat{M}^{*12} = 0 \; , \; \hat{Q}^{*12} = 0 \; , \; \hat{N}^{*1} = 0 \; . \qquad (6.60)$$

Next, taking the symmetry edge to be the $\xi^2 = \xi_o^2$ coordinate curve, we locate the symmetry plane at perpendicular distance K_2 from the origin and orient it relative to the basis $\underline{k}_i$ such that $\underline{k}_2$ is normal and hence $\underline{k}_\tau$ ($\tau = 1,3$) parallel to the plane. With the notation

$$\delta_+ = \delta \; (\xi^1, \; \xi_o^2 + \Delta\xi^2) \; , \quad \delta_- = \delta \; (\xi^1, \; \xi_o^2 - \Delta\xi^2) \; , \quad \delta = \delta \; (\xi^1, \; \xi_o^2) \; ,$$

we then have that the position vector satisfies

$$(\underline{r}_+ + \underline{r}_-) \cdot \underline{k}_2 = 2 \; \underline{r} \cdot \underline{k}_2 = 2 \; K_2 \; , \; (\underline{r}_+ - \underline{r}_-) \cdot \underline{k}_\tau = 0 \; (\tau = 1,3). \qquad (6.61)$$

As shown in Appendix B, with these relations we obtain the symmetry of the variables appearing in (5.42):

$$(\underline{a}_+^2 - \underline{a}_-^2) \cdot \underline{k}_2 = 0 \quad , \qquad (\underline{a}_+^2 + \underline{a}_-^2) \cdot \underline{k}_\tau = 0 \quad ,$$

$$(\underline{a}_+^1 + \underline{a}_-^1) \cdot \underline{k}_2 = 0 \quad , \qquad (\underline{a}_+^1 - \underline{a}_-^1) \cdot \underline{k}_\tau = 0 \quad , \qquad (6.62)$$

$$(\underline{n}_+ + \underline{n}_-) \cdot \underline{k} = 0 \quad , \qquad (\underline{n}_+ - \underline{n}_-) \cdot \underline{k}_\tau = 0 \quad ,$$

$$(\underset{\sim}{v}_+ + \underset{\sim}{v}_-) \cdot \underset{\sim}{k}_2 = 0 \quad , \qquad (\underset{\sim}{v}_+ - \underset{\sim}{v}_-) \cdot \underset{\sim}{k}_\tau = 0 \quad ,$$

$$(\underset{\sim}{\omega}_+ + \underset{\sim}{\omega}_-) \cdot \underset{\sim}{k}_2 = 0 \quad , \qquad (\underset{\sim}{\omega}_+ - \underset{\sim}{\omega}_-) \cdot \underset{\sim}{k}_\tau = 0 \quad , \qquad (6.63)$$

$$(\underset{\sim}{F}^*_+ + \underset{\sim}{F}^*_-) \cdot \underset{\sim}{k}_2 = 0 \quad , \qquad (\underset{\sim}{F}^*_+ - \underset{\sim}{F}^*_-) \cdot \underset{\sim}{k}_\tau = 0 \quad ,$$

$$\hat{M}^{*11}_+ = \hat{M}^{*11}_- \quad , \qquad \hat{M}^{*12}_+ = - \hat{M}^{*12}_- \quad , \qquad \hat{M}^{*22}_+ = \hat{M}^{*22}_- \quad ,$$

$$\hat{Q}^{*11}_+ = \hat{Q}^{*11}_- \quad , \qquad \hat{Q}^{*12}_+ = - \hat{Q}^{*12}_- \quad , \qquad \hat{Q}^{*22}_+ = \hat{Q}^{*22}_- \quad , \qquad (6.64)$$

$$\hat{N}^{*1}_+ = \hat{N}^{*1}_- \quad , \qquad \hat{N}^{*2}_+ = - \hat{N}^{*2}_- \quad .$$

When we take the limit as $\Delta\zeta^2 \to 0$, using the fact that at the boundary $\underset{\sim}{a}_2$ and $\underset{\sim}{a}^2$ are colinear with $\underset{\sim}{k}_2$, we find

$$v^2 = 0 \ , \ \omega^2 = 0 \ , \ F^{*2} = 0 \ , \ \hat{M}^{*12} = 0 \ , \ \hat{Q}^{*12} = 0, \ \hat{N}^{*2} = 0 \ . \quad (6.65)$$

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7. MATHEMATICAL FORMULATION OF THE REPSIL CODE

In this chapter we begin by specializing the general shell theory developed in the previous chapters to the theory on which the finite difference formulation used in REPSIL is based. This specialization is accomplished in two steps: first the thin shell approximation is applied to the general equation and then further restrictions, both physical and mathematical, are imposed to yield the special theory on which REPSIL is based. The chapter concludes with a description of how this theory is implemented in REPSIL and of the computational procedure used to obtain a solution.

7.1 Equations of Thin Shell Theory

The theory developed in Chapters 2 through 6 assumes only that the deformation undergone by the shell satisfies the Kirchhoff hypothesis. As noted on page 29, this assumption entails a mild restriction: the deformation must be such that the minimum principle radius of curvature is greater than half the thickness; i.e. $\frac{h}{2} < R_{min}$. Otherwise, the position of particles would not be unique.

We now impose a more stringent restriction on the deformation by requiring R_{min} be so much larger than h that terms of order $|\zeta\ b_\beta^\alpha|^2$ and higher can be neglected and dropped in comparison to unity. In other words, we assume that for sufficiently thin shells these higher order terms have little affect on the deformation. The resulting equations we call the *thin shell equations* and the resulting theory – *thin shell theory*. The theory developed here differs from the classical theory of thin shells, in that the latter neglects terms of order $|\zeta\ b_\beta^\alpha|$ in all but the expressions for the strain[*]. This difference will be pointed out in the development.

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[*] A clear and straightforward exposition of classical thin shell theory using the Kirchhoff hypothesis is given in [13; Ch. 2]. Direct comparison of the classical equations as set forth in this reference and the equations of the present work is however made difficult by the unnecessary use of principal curvature coordinates in the former.

We will be mainly concerned with obtaining the thin shell versions of the equations associated with the reduced equation of motion (5.45), which form the basis of the REPSIL formulation. The specialization to thin shells of equations associated with other forms of the equations of motion is however straightforward.

Relations involving the geometry of the middle surface will remain unchanged. Only geometric relations off the middle surface will be altered by the thin shell assumption, due to their dependence on $\zeta \, b_\beta^\alpha$ through the variable μ_β^α being made linear. Hence, while the metric $a_{\alpha\beta}$ is unchanged, the metric $g_{\alpha\beta}$ given by (2.46) becomes

$$g_{\alpha\beta} = a_{\alpha\beta} - 2\,\zeta\,b_{\alpha\beta} \,. \tag{7.1}$$

It follows from (3.12) that the incremental strain-displacement relation (3.20) also becomes linear in ζ:

$$\Delta E_{\alpha\beta} = \frac{1}{2}\left[(\underset{\sim}{a}_\alpha \cdot \Delta\underset{\sim}{u},_\beta + \underset{\sim}{a}_\beta \cdot \Delta\underset{\sim}{u},_\alpha - \Delta\underset{\sim}{u},_\alpha \cdot \Delta\underset{\sim}{u},_\beta) \right.$$
$$\left. - 2\zeta(\underset{\sim}{n} \cdot \Delta\underset{\sim}{u},_{\alpha\beta} + \Delta\underset{\sim}{n} \cdot \underset{\sim}{a}_{\alpha,\beta} - \Delta\underset{\sim}{n} \cdot \Delta\underset{\sim}{u},_{\alpha\beta})\right] \,. \tag{7.2}$$

For computational convenience, the REPSIL formulation, rather than use the last form of the strain-displacement relation, uses the equivalent but shorter form

$$\Delta E_{\alpha\beta} = \frac{1}{2}\left[(\underset{\sim}{a}_\alpha \cdot \Delta\underset{\sim}{u},_\beta + \underset{\sim}{a}_\beta \cdot \Delta\underset{\sim}{u},_\alpha - \Delta\underset{\sim}{u},_\alpha \cdot \Delta\underset{\sim}{u},_\beta) \right.$$
$$\left. - 2\zeta(\underset{\sim}{n}^- \cdot \Delta\underset{\sim}{u},_{\alpha\beta} + \Delta\underset{\sim}{n} \cdot \underset{\sim}{a}_{\alpha,\beta})\right] \,. \tag{7.3}$$

The surface normal-displacement relation (3.30) is unaffected by the thin shell assumption.

Since all the resultant relations of Chapter 4 are obtained by integration through the thickness, they are all altered. In particular for the zeroth, first and second resu; moments of mass density (4.15), we now have

$$\gamma_a = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \ (1 - \zeta \ b_\gamma^\gamma)(\zeta)^a \ d\zeta \qquad (a = 0, \ 1, \ 2) \ , \qquad (7.4)$$

which follows from the linearization of the determinant $(2.47)_1$. Using these relations, the momentum-velocity relations (4.14) are automatically appropriate for thin shells. The resultant relations for the external force and couple (4.36) become

$$F^\alpha = \int_{-\frac{h}{2}}^{\frac{h}{2}} [b^\alpha - 2\zeta \ b_\beta^{(\alpha} \ b^{\beta)}] \ d\zeta + [t_+^\alpha + t_-^\alpha - h \ (b_\beta^{(\alpha} \ t_+^{\beta)} - b_\beta^{(\alpha} \ t_-^{\beta)})] \ ,$$

$$F = \int_{-\frac{h}{2}}^{\frac{h}{2}} [1 - \zeta \ b_\beta^\beta] \ b^3 d\zeta + [t_+^3 + t_-^3 - \frac{h}{2} b_\beta^\beta \ (t_+^3 - t_-^3)] \ , \qquad (7.5)$$

$$C^\alpha = \int_{-\frac{h}{2}}^{\frac{h}{2}} [b^\alpha - 2\zeta \ b_\beta^{(\alpha} \ b^{\beta)}] \ \zeta \ d\zeta + \frac{h}{2} [t_+^\alpha - t_-^\alpha - h \ (b_\beta^{(\alpha} \ t_+^{\beta)} + b_\beta^{(\alpha} \ t_-^{\beta)})] \ .$$

The stress resultant and couple resultant relations (4.28) assume the thin shell forms:

$$Q^{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} [\sigma^{\alpha\beta} - 2\zeta \ b_\gamma^{(\beta} \ \sigma^{\gamma)c}] \ d\zeta \ ,$$

$$Q^\alpha = \int_{-\frac{h}{2}}^{\frac{h}{2}} [1 - \zeta \ b_\gamma^\gamma] \ \sigma^{3\alpha} \ d\zeta \ , \qquad (7.6)$$

$$M^{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} [\sigma^{\alpha\beta} - 2\zeta \ b_\gamma^{(\beta} \ \sigma^{\gamma)\alpha}] \ \zeta \ d\zeta \ .$$

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However, of more direct concern to the REPSIL formulation are the thin shell forms of the resultant relations for the modified membrane and bending components:

$$\hat{Q}^{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} [1 - \zeta\, b_{\gamma}^{\gamma}]\, \sigma^{\alpha\beta}\, d\zeta \, ,$$

$$\hat{M}^{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} [(1 - \zeta\, b_{\gamma}^{\gamma})\, \sigma^{\alpha\beta} - \zeta\, b_{\gamma}^{(\alpha}\, \sigma^{\gamma\beta)}]\, \zeta\, d\zeta \, ,$$

(7.7)

which follow from (5.43). With the above thin shell expressions for γ_0, γ_1, γ_2, F^{α}, F, C^{α}, $\hat{Q}^{\alpha\beta}$ and $\hat{M}^{\alpha\beta}$, the reduced equation of motion (5.41) or (5.45) is appropriate for thin shell theory without need of change.

As mentioned before, classical thin shell theory goes further by dropping the linear terms in (7.1), (7.4) - (7.7), giving us the approximations:

$$g_{\alpha\beta} = a_{\alpha\beta} \, , \qquad \gamma_a = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho\, (\zeta)^a\, d\zeta \qquad (a = 0,\, 1,\, 2) \, ,$$

$$F^{\alpha} = \int_{-\frac{h}{2}}^{\frac{h}{2}} b^{\alpha}\, d\zeta + t_{+}^{\alpha} + t_{-}^{\alpha} \, , \qquad F = \int_{-\frac{h}{2}}^{\frac{h}{2}} b^3\, d\zeta + t_{+}^3 + t_{-}^3 \, ,$$

(7.8)

$$Q^{\alpha\beta} \equiv \hat{Q}^{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{\alpha\beta}\, d\zeta \, , \qquad Q^{\alpha} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{3\alpha}\, d\zeta \, ,$$

$$C^{\alpha} = \int_{-\frac{h}{2}}^{\frac{h}{2}} b^{\alpha}\, \zeta\, d\zeta + \frac{h}{2}\, (t_{+}^{\alpha} - t_{-}^{\alpha}) \, , \qquad M^{\alpha\beta} \equiv \hat{M}^{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{\alpha\beta}\, \zeta\, d\zeta \, .$$

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Notice that the distinction between components $Q^{\alpha\beta}$ & $M^{\alpha\beta}$ and the modified components $\hat{Q}^{\alpha\beta}$ & $\hat{M}^{\alpha\beta}$ disappears completely in classical thin shell theory. Moreover notice that if the linear terms in the strain expressions (7.2) & (7.3) were also dropped, then the strain would not take into account bending and so would be appropriate for membrane theory[*]. In any event, the theory forming the basis for the REPSIL formulation retains for the sake of consistancy terms linear in $\zeta\, b^{\alpha}_{\beta}$ in the membrane and bending components.

7.2 Restricted Thin Shell Theory Used by REPSIL

The equations on which the finite difference scheme used in REPSIL is based are simplified forms of the thin shell equations just developed. The simplifications are of two types: the first arise from the specific type of problems that REPSIL is designed to treat and the effects predominating therein, with no mathematical approximation being used. The second follows from the mathematical need to make the equations simpler to solve, and involve approximations. Also, we adopt an elastoplastic constitutive relation between stress and strain to make the formulation complete.

REPSIL is formulated to treat the large transient response of thin shells subjected to blast loads. For this reason the surface tractions are assumed to be predominantly due to blast pressures, with the effects of viscous drag being nil. Moreover, the effect of gravity through the body forces on the deformation is negligible compared to the blast pressure effect. Hence, the traction and body force fields have the special forms:

$$\underset{\sim}{t}_{+} = -\, P_{+}\, \underset{\sim}{n} \quad, \quad \underset{\sim}{t}_{-} = P_{-}\, \underset{\sim}{n} \quad, \quad \underset{\sim}{b} \equiv 0 \ , \tag{7.9}$$

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[*] Membrane theory is treated in [12; Sect. 14.3] and [13; Ch. 4].

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which when applied to the components of $\underset{\sim}{F}$ and $\underset{\sim}{C}$ as given by (7.5) yield

$$\underset{\sim}{F} = - [P_+ - P_- - \frac{h}{2} b^\beta_\beta (P_+ + P_-)] \underset{\sim}{n} \quad , \quad \underset{\sim}{C} = 0 \ . \qquad (7.10)$$

The shell is assumed to be initially homogeneous, so that the initial mass density ρ_o is constant. Hence, expressions (7.4) for the resultant moments of mass density can be integrated for their initial values, which gives

$$\Gamma_0 = \rho_c h \quad , \qquad \Gamma_1 = - \frac{1}{12} B^\gamma_\gamma \rho_o h^3 \quad , \qquad \Gamma_2 = \frac{1}{12} \rho_o h^3 . \quad (7.11)$$

Consequently, from (5.20) we have

$$\gamma^*_0 = \rho_o h A^{\frac{1}{2}} \ , \quad \gamma^*_1 = - \frac{1}{12} B^\gamma_\gamma \rho_o h^3 A^{\frac{1}{2}} \ , \quad \gamma^*_2 = \frac{1}{12} \rho_o h^3 A^{\frac{1}{2}}. (7.12)$$

As mentioned earlier, the first resultant moment of mass density is a measure of the distance from the middle surface to the center of mass for the section. It is clear from $(7.11)_2$ that for an *initially uniform thin* shell, the center of mass will coincide with the middle surface when the initial geometry of the shell is such that the mean curvature B^γ_γ vanishes, which is tantamount to the principal radii of curvature being equal but oppositely directed. It then follows that the first resultant moment of mass density will vanish for all subsequent time, even though the mean curvature b^γ_γ may no longer vanish. In particular this holds true for initially flat shells.

Expressions (7.10) - (7.12) are exact within thin shell theory. For the sake of simplicity, we now go further and assume that the first order terms can also be neglected in these expressions:

$$\underset{\sim}{F} = - [P_+ - P_-] \underset{\sim}{n} = - \Delta P \underset{\sim}{n} \quad , \qquad \underset{\sim}{C} = 0 \ ,$$

$$\Gamma_0 = \rho_o h \quad , \qquad \Gamma_1 = 0 \quad , \qquad \Gamma_2 = \frac{1}{12} \rho_o h^3 \ , \qquad (7.13)$$

$$\gamma^*_0 = \rho_o h A^{\frac{1}{2}} \quad , \qquad \gamma^*_1 = 0 \quad , \qquad \gamma^*_2 = \frac{1}{12} \rho_o h^3 A^{\frac{1}{2}} \ ,$$

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giving the classical thin shell approximation. In addition, an approximation is made that gives an explicit finite difference scheme. The $[\gamma^*_2 \, \underset{\sim}{a}{}^\alpha \cdot \underset{\sim}{\dot{\omega}} \, \underset{\sim}{n}]_{,\alpha}$ term in the reduced equation of motion (5.45) is assumed negligible in comparison to the $\gamma^*_0 \, \underset{\sim}{\dot{v}}$ term. Since by virtue of (2.31) $\underset{\sim}{\omega}$ is like a gradient of the velocity $\underset{\sim}{v}$, it follows from (7.13)$_{7,9}$ that this approximation is equivalent to neglecting $\left| h^2 \, \nabla^2 \, \underset{\sim}{\dot{v}} \right|$ in comparison to $\left| \underset{\sim}{\dot{v}} \right|$. In other words, the wave lengths of acceleration waves are assumed long compared to the shell thickness. With the last approximation it is to be expected that the solution using this theory will lose accuracy whenever higher harmonics of the shell are significantly excited.

The above specializations of the thin shell equations affect only the equations of motion. In particular, the reduced equation of motion (5.45) becomes

$$\gamma^*_0 \, \underset{\sim}{\dot{v}} \; = \hat{\underset{\sim}{M}}{}^{*\alpha\beta}{}_{,\alpha\beta} + \hat{\underset{\sim}{N}}{}^{*\alpha}{}_{,\alpha} + \underset{\sim}{F}{}^* \tag{7.14}$$

with

$$\gamma^*_0 = \rho_0 \, h \, A^{\frac{1}{2}} \;, \qquad \underset{\sim}{F}{}^* = - \, \Delta P \, a^{\frac{1}{2}} \, \underset{\sim}{n}$$

$$\hat{\underset{\sim}{M}}{}^{*\alpha\beta} = \hat{M}{}^{*\alpha\beta} \, \underset{\sim}{n} \;, \qquad \hat{\underset{\sim}{N}}{}^{*\alpha} = \hat{Q}{}^{*\alpha\beta} \underset{\sim}{a}_\beta + \Gamma_{\beta\gamma}{}^\alpha \, \hat{M}{}^{*\beta\gamma} \, \underset{\sim}{n} \;. \tag{7.15}$$

The membrane and bending components are not specialized, but follow from the thin shell expressions (7.7):

$$\hat{Q}{}^{*\alpha\beta} = a^{\frac{1}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} (1 - \zeta \, b^\gamma_\gamma) \, \sigma^{\alpha\beta} \, d\zeta$$

$$\tag{7.16}$$

$$\hat{M}{}^{*\alpha\beta} = a^{\frac{1}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} [(1 - \zeta \, b^\gamma_\gamma) \, \sigma^{\alpha\beta} - \zeta \, b^{(\alpha}_\gamma \, \sigma^{\gamma\beta)}] \, \zeta \, d\zeta \;.$$

Defining the zeroth, first and second moments of stress as

$$\Sigma_a^{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{\alpha\beta} (\zeta)^a \, d\zeta \qquad (a = 0, 1, 2) \;, \qquad (7.17)$$

the last two relations can be expressed as:

$$\hat{Q}\!\star^{\alpha\beta} = a^{\frac{1}{2}} [\Sigma_0^{\alpha\beta} - b_\gamma^\gamma \, \Sigma_1^{\alpha\beta}] \;,$$

$$\hat{M}\!\star^{\alpha\beta} = a^{\frac{1}{2}} [\Sigma_1^{\alpha\beta} - b_\gamma^\gamma \, \Sigma_2^{\alpha\beta} - b^{(\alpha}_{\ \gamma} \, \Sigma_2^{\gamma\beta)}] \;. \qquad (7.18)$$

In order to complete the formulation it is necessary to specify in what way the stress is related to the strain. Before doing this, we first characterize the stress field by assuming that a state of plane stress exists in each lamella parallel to the middle surface and that the shell is composed of isotropic material. However, for an isotropic material the strain components given by the Kirchhoff hypothesis, see equation (3.6), are inconsistant with the plane stress assumption, since the vanishing normal strain component will usually result in a non-zero normal stress component. This situation is resolved in the usual way by using the Kirchhoff hypothesis to determine only the tangential components of strain, with the normal components following from the plane stress conditions.

Presently, the REPSIL formulation employs an elastoplastic stress-strain law that is linear in the elastic range and is capable of modelling strain hardening and strain rate effects through the use of the mechanical sublayer model[*]. Using the above assumptions we derive in Appendix C an expression relating the incremental change in the stress components to the components of the strain increment and the present values of the stress components:

$$\Delta\sigma_\beta^\alpha = \frac{E}{1+\nu} [\Delta\varepsilon_\beta^\alpha + \frac{\nu}{1-\nu} \delta_\beta^\alpha \, \Delta\varepsilon_\gamma^\gamma] - \Delta\lambda \, [\sigma_\beta^\alpha - \frac{1-2\nu}{3(1-\nu)} \delta_\beta^\alpha \, \sigma_\gamma^\gamma] \;, \qquad (7.19)$$

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[*] See [2; Sect. 5.4.2] for a concise description of the mechanical sublayer concept.

wherein the mixed components are with respect to the basis $g_{\alpha\beta}$:

$$\Delta\sigma^{\alpha}_{\beta} = g_{\beta\gamma} \, \Delta\sigma^{\gamma\alpha} \quad , \qquad \Delta E^{\alpha}_{\beta} = g^{\alpha\gamma} \, \Delta E_{\gamma\beta} \quad , \qquad \sigma^{\alpha}_{\beta} = g_{\beta\gamma} \, \sigma^{\gamma\alpha} \quad . \quad (7.20)$$

The factor $\Delta\lambda$ is non-negative and its value is determined from the von Mises-Hencky yield condition:

$$\phi(\sigma^{\alpha}_{\beta}) \equiv \sigma^{\alpha}_{\beta} \, \sigma^{\beta}_{\alpha} - \frac{1}{3} \, (\sigma^{\gamma}_{\gamma})^2 - \frac{2}{3} \, \sigma_{o}^{\,2} \leqslant 0 \quad , \qquad\qquad (7.21)$$

with σ_o the uniaxial yield stress. When $\Delta\lambda = 0$ in (7.19) results in a state of stress $\sigma^{\alpha}_{\beta} + \Delta\sigma^{\alpha}_{\beta}$ outside the yield surface $\phi(\sigma^{\alpha}_{\beta} + \Delta\sigma^{\alpha}_{\beta}) > 0$, then the proper value of $\Delta\lambda$ is the smallest positive value that satisfies $\phi(\sigma^{\alpha}_{\beta} + \Delta\sigma^{\alpha}_{\beta}) = 0$. Otherwise, $\Delta\lambda = 0$ is the proper value since it gives a state of stress on or within the yield surface $\phi(\sigma^{\alpha}_{\beta} + \Delta\sigma^{\alpha}_{\beta}) \leqslant 0$. The value of $\Delta\lambda$ is thus determined by (7.21) and the stress increments (7.19) are completely specified. For a strain hardening material the stress components and yield stress pertain to each mechanical sublayer. A procedure for dealing with the possibility of complex $\Delta\lambda$ has been developed in [4] and is presently incorporated in the REPSIL code.

7.3 Description of the REPSIL Formulation

We give a description of the computational procedure employed in the REPSIL code, including an outline of the algorithm used to advance the solution one time step*. The computational procedure is similar to that used in PETROS 1 and 2, so that the following description will facilitate the comparison of these codes in the next chapter.

Basically all these codes solve finite difference forms of partial differential equations that characterize the transient deformation of a thin Kirchhoff shell. Finite differencing with respect to the material coordinates ξ^{α} results in the dependent variables of the theory being defined on a two dimensional Lagrangian mesh or grid of points. At

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*A detailed description of the REPSIL code and its computational formulation will be soon forthcoming in a user's manual.

interior mesh points and at those along symmetry boundaries central finite difference operators are employed, while along fixed boundaries (either hinged as clamped) forward or backward difference operators are used, with all operators being of order $\left|\Delta\xi^{\alpha}\right|^{2*}$. At present the Lagrangian mesh has a rectangular boundary, which means that the REPSiL formulation is restricted to shells having simply connected middle surfaces bounded by four smooth arcs.

The shell is conceptually divided by a number of lamellae parallel to the middle surface into K layers within which the strain components are assumed constant at their mid-layer values, making the stress components similarly uniform. The reason this is done, despite the fact that the strain increments are simple linear functions of ζ as can be seen from (7.3), is that plastic flow soon makes impossible the simple analytic characterization of the variation of the stress components through the thickness. Thus, the components of stress and strain are defined at stations $k = 1,\ldots K$ through the thickness, as well as at each point of the Lagrangian mesh. This dependence on k will not be explicitly indicated in what follows.

Replacement of time derivatives in the shell equation by their equivalent finite difference operators results in an explicit finite difference scheme characterizing the manner in which the dependent variables of the shell change during a time interval Δt. Hence knowing the values of the shell variable at time t, we are able to calculate their values at the time $t + \Delta t$. The finite time derivatives employed are centered and of order $\left|\Delta t\right|^{2}$. The interval Δt does not change with time, but is fixed at a value for which the explicit finite difference scheme is stable.

Before a calculation can begin, the material properties and the initial geometry of the shell must be specified. An initial velocity and/or initial pressure distribution starts the calculation, giving the

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* A finite difference formulation of the free edge conditions has yet to be added to the REPSIL code.

initial change that the shell variables undergo. After that a sequence
of calculations are repeated cyclically, using when appropriate
pressure-time data, to generate later and later values of the shell
variables progressively until some prescribed maximum time is reached,
at which calculations terminate.

With the above brief description of the finite difference
formulation and computational procedure employed in the REPSIL code in
mind, we are ready to describe the algorithm that forms the heart of the
sequential computations advancing the solution one time step[*]. To keep
this description simple we shall not treat a strain hardening or strain
rate sensitive material, although as mentioned earlier REPSIL can treat
materials exhibiting such phenomena. The algorithm is a recipe for
solving the set of algebraic equations that result from finite
differencing the shell equations. For our purposes, it is sufficient
to exhibit the finite time derivative explicitly, continuing to
symbolize the difference operators with respect to ξ^{α} by partial
derivatives[**].

Let us assume that calculations have progressed n-1 time steps to
the time $t^- = (n-1)\,\Delta t$ and that the position vector r and normal vector
n are known at each mesh point and the stress components $\sigma^{\alpha\beta}$ are known
at each mesh point and station through the thickness. Also the
displacement increment Δu for the time interval $t^- = (n-1)\,\Delta t$ to
$t = n\,\Delta t$ is assumed to have just been calculated at each mesh point.
Using the notation

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[*] A less detailed description is presented in a recent report
[5; Sect. II].

[**] The precise finite difference expressions will be in the forthcoming
user's manual.

$$\zeta^- = \zeta\,[(n-1)\Delta t] \quad , \qquad \zeta = \zeta\,[n\Delta t] \quad , \qquad \zeta^+ = \zeta\,[(n+1)\Delta t] \quad ,$$

$$\Delta\zeta = \zeta - \zeta^- \quad , \qquad \Delta\zeta^+ = \zeta^+ - \zeta \,,^{\textstyle *}$$

our objective is to outline the sequence of calculations by which the new values r, n, $\sigma^{\alpha\beta}$, Δu^+ of the shell variables are generated from the previously known values r^-, n^-, $\sigma^{-\alpha\beta}$, Δu. This sequence basically involves five calculations, performed in the following order:

7.3.1 **Current Geometry Calculation.** The current position vector r is found from (3.11):

$$\underset{\sim}{r} = \underset{\sim}{r}^- + \underset{\sim}{\Delta u} \, . \tag{7.22}$$

Finite differencing the current position vector r gives the variables characterizing the differential geometry of the middle surface (see the equations of section 2.2):

$$\underset{\sim}{a}_\alpha = \underset{\sim}{r},_\alpha \quad , \qquad a_{\alpha\beta} = \underset{\sim}{a}_\alpha \cdot \underset{\sim}{a}_\beta \,,$$

$$a = a_{11}\, a_{22} - (a_{12})^2 \quad , \qquad \underset{\sim}{n} = \underset{\sim}{a}_1 \times \underset{\sim}{a}_2 / a^{\frac{1}{2}} \,,$$

$$a^{11} = a_{22}/a \quad , \quad a^{12} = -\,a_{12}/a \quad , \qquad a^{22} = a_{11}/a \,, \tag{7.23}$$

$$b_{\alpha\beta} = \underset{\sim}{n} \cdot \underset{\sim}{r},_{\alpha\beta} \quad , \qquad b_\alpha^\beta = a^{\beta\gamma}\, b_{\gamma\alpha} \,,$$

$$\underset{\sim}{a}^\alpha = a^{\alpha\beta}\, \underset{\sim}{a}_\beta \quad , \qquad \Gamma_{\alpha\beta}{}^\gamma = \underset{\sim}{a}^\gamma \cdot \underset{\sim}{r},_{\alpha\beta} \, .$$

7.3.2 **Strain Increment Calculation.** In order to determine the strain increments, it is first necessary to find the incremental change in the surface normal by finite differencing the displacement increments Δu and using (3.30) and (3.31):

$$\Delta n_\alpha = -\,\underset{\sim}{n}^- \cdot \underset{\sim}{\Delta u},_\alpha \quad , \qquad \underset{\sim}{\Delta n} = [\underset{\sim}{a}_\alpha + \frac{\Delta n_\alpha}{1+\underset{\sim}{n}\cdot\underset{\sim}{n}^-}\,\underset{\sim}{n}]\, a^{\alpha\beta}\, \Delta n_\beta \, . \tag{7.24}$$

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*Notice that this notation is in accord with that employed earlier in Sections 3.2 and 3.3.

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The strain increments through the thickness at stations $\zeta(k)$, $k = 1,\ldots K$, are calculated from the thin shell relation (7.3):

$$\Delta E_{\alpha\beta} = \frac{1}{2} \left(\underset{\sim}{a}_\alpha \cdot \Delta u,_\beta + \underset{\sim}{a}_\beta \cdot \Delta u,_\alpha - \Delta u,_\alpha \cdot \Delta u,_\beta \right.$$
$$\left. - \zeta \left(\underset{\sim}{n}^- \cdot \Delta u,_{\alpha\beta} + \Delta n \cdot \underset{\sim}{r},_{\alpha\beta} \right) \right. . \tag{7.25}$$

Notice that the formulae for Δn and $\Delta E_{\alpha\beta}$ are at least linear in Δu, assuring numerical accuracy by circumventing the need to take differences of nearly equal terms.

7.3.3 Elastoplastic Stress Calculation.

The stress calculation uses mixed components of stress and strain. Hence, it is necessary to know the metric $g_{\alpha\beta}$ and inverse $g^{\alpha\beta}$ at stations $\zeta(k)$ through the thickness. These follow from the thin shell relation (7.1) and the equations of section 2.3:

$$g_{\alpha\beta} = a_{\alpha\beta} - 2\zeta \, b_{\alpha\beta} \quad , \qquad g = g_{11} \, g_{22} - (g_{12})^2 \quad ,$$
$$g^{11} = g_{22}/g \quad , \qquad g^{12} = -g_{12}/g \quad , \qquad g^{22} = g_{11}/g \quad . \tag{7.26}$$

With these relations the mixed components of the strain increment and the stress are obtained:

$$\Delta E^\alpha_\beta = g^{\alpha\gamma} \, \Delta E_{\gamma\beta} \quad , \qquad \sigma^{-\alpha}_{\ \beta} = g_{\beta\gamma} \, \sigma^{-\gamma\alpha} \quad . \tag{7.27}$$

Next, the components of the elastic stress increment

$$\overset{E}{\Delta\sigma}{}^\alpha_\beta = \frac{E}{1+\nu} \left[\Delta E^\alpha_\beta + \frac{\nu}{1-\nu} \Delta E^\gamma_\gamma \, \delta^\alpha_\beta \right] \tag{7.28}$$

and the plastic flow corrector stress

$$\overset{C}{\sigma}{}^\alpha_\beta = \sigma^{-\alpha}_{\ \beta} \, \frac{1-2\nu}{3(1-\nu)} \, \sigma^{-\gamma}_{\ \gamma} \, \delta^\alpha_\beta \tag{7.29}$$

are determined and used to find the current values of the stress components using (7.19):

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$$\sigma^{\alpha}_{\beta} = \overset{T}{\sigma}{}^{\alpha}_{\beta} - \Delta\lambda\overset{C}{\sigma}{}^{\alpha}_{\beta} \; , \tag{7.30}$$

with

$$\overset{T}{\sigma}{}^{\alpha}_{\beta} = \sigma^{-\alpha}_{\beta} + \Delta\overset{E}{\sigma}{}^{\alpha}_{\beta} \; . \tag{7.31}$$

The value of $\Delta\lambda$ depends on $\overset{T}{\sigma}{}^{\alpha}_{\beta}$ through the yield condition (7.21). When $\overset{T}{\sigma}{}^{\alpha}_{\beta}$ is on or within the yield surface

$$\overset{T}{\sigma}{}^{\alpha}_{\beta} \overset{T}{\sigma}{}^{\beta}_{\alpha} - \frac{1}{3} (\overset{T}{\sigma}{}^{\gamma}_{\gamma})^2 \leqslant \frac{2}{3} \sigma^2_0 \; , \tag{7.32}$$

then $\Delta\lambda = 0$ and current stress components are $\overset{T}{\sigma}{}^{\alpha}_{\beta}$; otherwise, $\Delta\lambda$ is the smallest positive value giving a current stress on the yield surface

$$\sigma^{\alpha}_{\beta} \sigma^{\beta}_{\alpha} - \frac{1}{3} (\sigma^{\gamma}_{\gamma})^2 = \frac{2}{3} \sigma^2_0 \; . \tag{7.33}$$

In either event a value for $\Delta\lambda$ and hence the values of σ^{α}_{β} are determined. Finally the contravariant components of the stress are found

$$\sigma^{\alpha\beta} = g^{\alpha\gamma} \sigma^{\beta}_{\gamma} \; . \tag{7.34}$$

When the value of $\Delta\lambda$ is complex, a procedure developed by HUFFINGTON [4] is used to determine the stress. As pointed out in Appendix C, the stress calculation is not exactly centered with respect to time due to the use of $\sigma^{-\alpha}_{\beta}$ to calculate $\overset{C}{\sigma}{}^{\alpha}_{\beta}$.

7.3.4 Membrane and Bending Resultants Calculation.

The zeroth, first and second moment of stress are determined using (7.17)

$$\Sigma^{\alpha\beta}_a = \sum_k \sigma^{\alpha\beta} (\zeta)^a \Delta\zeta \qquad (a = 0, 1, 2) \tag{7.35}$$

wherein the summation sign over k is used to emphasize that numerical integration through the shell thickness is involved. With the above values of stress moments, the membrane and bending components follow

from the thin shell relations (7.18):

$$\hat{Q}*^{\alpha\beta} = a^{\frac{1}{2}} [\Sigma_0^{\alpha\beta} - b_\gamma^\gamma \Sigma_1^{\alpha\beta}] ,$$

$$\hat{M}*^{\alpha\beta} = a^{\frac{1}{2}} [\Sigma_1^{\alpha\beta} - b_\gamma^\gamma \Sigma_2^{\alpha\beta} - b_\gamma^{(\alpha} \Sigma_2^{\gamma\beta)}] . \tag{7.36}$$

7.3.5 Equation of Motion Calculation.

Using the just determined values of $\hat{Q}*^{\alpha\beta}$ and $\hat{M}*^{\alpha\beta}$ and the a priori specified values of the pressure differential ΔP between the shell faces, the vectors

$$\tilde{\hat{N}}*^\alpha = \hat{Q}*^{\alpha\beta} \underset{\sim}{a}_\beta + \Gamma_{\beta\gamma}{}^\alpha \tilde{\hat{M}}*^{\beta\gamma} \underset{\sim}{n} ,$$

$$\tilde{\hat{M}}*^{\alpha\beta} = \hat{M}*^{\alpha\beta} \underset{\sim}{n} , \qquad \underset{\sim}{F}* = - \Delta P a^{\frac{1}{2}} \underset{\sim}{n} \tag{7.37}$$

are determined (cf. (7.15)). The finite difference form of the equation of motion (7.14) with the above values gives the displacement increment $\Delta\underset{\sim}{u}^+$ for the next time increment:

$$\Delta\underset{\sim}{u}^+ = \Delta\underset{\sim}{u} + \frac{(\Delta t)^2}{\gamma^*_0} [\tilde{\hat{M}}*^{\alpha\beta}{}_{,\alpha\beta} + \tilde{\hat{N}}*^\alpha{}_{,\alpha} + \underset{\sim}{F}*] , \tag{7.38}$$

wherein γ^*_0 is the initially determined time constant

$$\gamma^*_0 = \rho_0 h A^{\frac{1}{2}} . \tag{7.39}$$

This calculation completes the cycle, the current values $\underset{\sim}{r}$, $\underset{\sim}{n}$, $\sigma^{\alpha\beta}$, $\Delta\underset{\sim}{u}^+$ having been determined from the previous values $\overset{-}{\underset{\sim}{r}}$, $\overset{=}{\underset{\sim}{n}}$, $\overset{\approx}{\sigma}^{\alpha\beta}$, $\Delta\underset{\sim}{u}$. Hence, the solution has been advanced a time step from $\overset{\approx}{t} = (n-1) \Delta t$ to $t = n\Delta t$. Repeating this sequence, the computer code generates later and later values of $\underset{\sim}{r}$, $\underset{\sim}{n}$, $\sigma^{\alpha\beta}$, $\Delta\underset{\sim}{u}$ until some prescribed maximum time $t_{max} = N\Delta t$ is reached.

7.3.6 Energy Balance and Strain Calculations.

The sequence of calculations just outlined is vital for the advancement of the solution. Concurrent with this sequence there are performed calculations that, although not essential to the solution, do provide information about the current state of the shell. More specifically, as a global measure of

the deformation, the work done by the pressure, the kinetic energy and
the elastic strain energy are calculated for each time cycle. The
difference between the pressure work and the sum of the kinetic and strain
energy indicates the amount of energy degradation caused by plastic flow.
Also, as a local measure of the deformation, the physical components of
strain at selected points on the bounding surfaces of the shell are
calculated for each time cycle. These components are exact for large
strains. The equations employed in the REPSIL code for the strain
calculations are derived in Appendix D of this report. The derivation
of the equations used for the energy calculations is reserved for a
subsequence report. The forthcoming user's manual will describe how
both the energy balance and strain equations are implemented in the
REPSIL code.

8. COMPARISON OF THE REPSIL, PETROS 2 AND PETROS 1 CODES

As mentioned in the introductory chapter, the REPSIL and PETROS 2 codes are developments of the PETROS 1 code, both correcting and generalizing the PETROS 1 formulation. In this chapter we compare the theoretical and computational capabilities of these codes. First the theoretical formulations used in these codes are compared and it is shown how the theory on which PEiROS 1 is based has been generalized and made less restrictive. Next the computational improvements and increased capabilities of REPSIL and PETROS 2 over PETROS 1 are qualitatively compared. The results of these comparisons are summarized in tables and the implications of the differences between these codes are discussed briefly.

8.1 Comparison of Theoretical Formulations

The REPSIL, PETROS 1 and PETROS 2 codes are based on theories that are quite similar. All assume the Kirchhoff hypothesis to hold and all restrict themselves to thin shells. Often the formulations of these codes are based on the same equations. Essentially all employ the same formulae to describe the differential geometry of the middle surface. They differ principally in the equations on which $1°$ the strain increment calculation, $2°$ the elastoplastic stress calculation, $3°$ the membrane and bending resultant calculation and $4°$ the associated equation of motion calculation are based (cf. Sections 7.3.2 - 7.3.5). We will discuss these differences in this order, combining the last two.

8.1.1 Strain increment Formulation. The various strain-displacement relations employed in these codes have as common bases the Kirchhoff hypothesis and thin shell assumption. The precise forms of the relations can be summarized as follows:

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REPSIL (cf. (7.24) & (7.25)):

$$\Delta E_{\alpha\beta} = \frac{1}{2} (a_{\alpha} \cdot \Delta u,_{\beta} + a_{\beta} \cdot \Delta u,_{\alpha} - \Delta u,_{\alpha} \cdot \Delta u,_{\beta}$$

$$- \zeta(\bar{n} \cdot \Delta u,_{\alpha\beta} + \Delta n \cdot r,_{\alpha}) , \tag{8.1}$$

$$\Delta n_{\alpha} = - \bar{n} \cdot \Delta u,_{\alpha} , \qquad \Delta n = \frac{a^{\alpha\beta} \Delta n_{\alpha} \Delta n_{\beta}}{1 + n \cdot \bar{n}} ,$$

PETROS 2 [2; Eq. 6.18, 5.23 & 5.29)]:

$$\Delta E_{\alpha\beta} = \frac{1}{2} (a_{\alpha} \cdot \Delta u,_{\beta} + a_{\beta} \cdot \Delta u,_{\alpha} - \Delta u,_{\alpha} \cdot \Delta u,_{\beta})$$

$$- \zeta(n \cdot \Delta u,_{\alpha\beta} + \Delta n \cdot r,_{\alpha\beta} - \Delta n \cdot \Delta u,_{\alpha\beta}) , \tag{8.2}$$

$$\Delta n_{\alpha} = - \bar{n} \cdot \Delta u,_{\alpha} , \qquad \Delta n = 1 - (1 - a^{\alpha\beta} \Delta n_{\alpha} \Delta n_{\beta})^{\frac{1}{2}} ,$$

PETROS 1 [1; Eq. 3.46 & 3.40]:

$$\Delta E_{\alpha\beta} = \frac{1}{2} [(a_{\alpha} \cdot \Delta u,_{\beta} + a_{\beta} \cdot \Delta u,_{\alpha}) + \zeta(a_{\alpha} \cdot \Delta n,_{\beta} + a_{\beta} \cdot \Delta n,_{\alpha})$$

$$\Delta n_{\alpha} = - n \cdot \Delta u,_{\alpha} - n,_{\alpha} \cdot \Delta u , \qquad \Delta n = 0 , \tag{8.3}$$

with Δn_{α} & Δn being the components of Δn for all three formulations:

$$\Delta n = \Delta n_{\alpha} a^{\alpha} + \Delta n \, n . \tag{3.4}$$

From $(3.16)_2$ it is clear that PETROS 2 uses an expression for $\Delta E_{\alpha\beta}$ that is exactly equivalent to that used by REPSIL. Also, both use the same expression for Δn_{α}. The substitution of $(3.16)_2$ in $(8.1)_3$ gives a quadratic equation on Δn of which $(8.2)_3$ is the solution under the mild condition $n \cdot \bar{n} \geqslant 0$, so that both REPSIL and PETROS 2 use equivalent expressions for Δn. Hence, REPSIL and PETROS 2 have equivalent strain increment formulations.

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In order to compare the expression for $\Delta E_{\alpha\beta}$ used in PETROS 1, we need to commute the order of differentiating terms in $(8.1)_1$ using (3.24) to obtain the

$$\Delta E_{\alpha\beta} = \frac{1}{2} [(\underset{\sim}{a}_\alpha \cdot \Delta \underset{\sim}{u},_\beta + \underset{\sim}{a}_\beta \cdot \Delta \underset{\sim}{u},_\alpha - \Delta \underset{\sim}{u},_\alpha \cdot \Delta \underset{\sim}{u},_\beta)$$

$$+ \zeta(\underset{\sim}{a}_\alpha \cdot \Delta \underset{\sim}{n},_\beta + \underset{\sim}{a}_\beta \cdot \Delta \underset{\sim}{n},_\alpha + \underset{\sim}{n},_\alpha \cdot \Delta \underset{\sim}{u},_\beta + \underset{\sim}{n},_\beta \cdot \Delta \underset{\sim}{u},_\alpha \qquad (8.5)$$

$$- \Delta \underset{\sim}{n},_\alpha \cdot \Delta \underset{\sim}{u},_\beta - \Delta \underset{\sim}{n},_\beta \cdot \Delta \underset{\sim}{u},_\alpha)]^*$$

When this expression and the REPSIL expressions for the components of $\Delta \underset{\sim}{n}$ are linearized by neglecting products of gradients of the displacement increments, there result the equations

$$\Delta E_{\alpha\beta} = \frac{1}{2} [(\underset{\sim}{a}_\alpha \cdot \Delta \underset{\sim}{u},_\beta + \underset{\sim}{a}_\beta \cdot \Delta \underset{\sim}{u},_\alpha)$$

$$+ \zeta(\underset{\sim}{a}_\alpha \cdot \Delta \underset{\sim}{n},_\beta + \underset{\sim}{a}_\beta \cdot \Delta \underset{\sim}{n},_\alpha + \underset{\sim}{n},_\alpha \cdot \Delta \underset{\sim}{u},_\beta + \underset{\sim}{n},_\beta \cdot \Delta \underset{\sim}{u},_\alpha)] \;,$$

$$(8.6)$$

$$\Delta n_\alpha = - \underset{\sim}{n} \cdot \Delta \underset{\sim}{u},_\alpha \;, \qquad \Delta n = 0 \;.$$

These equations would coincide with the PETROS 1 expressions for $\Delta E_{\alpha\beta}$, Δn_α & Δn if the term $\underset{\sim}{n},_\alpha$ were negligible, which means that the middle surface would have to remain virtually flat. Hence, PETROS 1 uses an approximation to the exact REPSIL strain increment formulation which is valid only when displacement gradients are small and the shell is virtually flat throughout the deformation. Consequently, for the particular shell geometry treated by PETROS 1, a cylindrical panel, it cannot be expected to give good results.

8.1.2 Elastoplastic Stress Formulation. Both the REPSIL and PETROS 2 stress formulations account for the variation in the metric through the thickness, while the PETROS 1 formulation disregards this variation. Specifically, these codes use the following formulae for the metric and inverse:

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* Notice that this expression is equivalent to the REPSIL expression for $\Delta E_{\alpha\beta}$ in the partial differential sense, but not the finite difference sense.

REPSIL (cf. (7.26))

$$g_{\alpha\beta} = a_{\alpha\beta} - 2\zeta\, b_{\alpha\beta} \quad , \quad g = g_{11}\, g_{22} - (g_{12})^2$$

$$g^{11} = g_{22}/g \;\; , \quad g^{12} = -\, g_{12}/g \quad , \quad g^{22} = g_{11}/g \;\; ,$$

(8.7)

PETROS 2 [2; Eq 5.106 & 5.107]

$$g_{\alpha\beta} = a_{\alpha\beta} - 2\zeta\, b_{\alpha\beta} \quad , \quad g^{\alpha\beta} = a^{\alpha\beta} + 2\zeta\, b^{\alpha\beta} \quad , \tag{8.8}$$

PETROS 1 [1; Sect. 5.1.2]

$$g_{\alpha\beta} = A_{\alpha\beta} \equiv \delta_{\alpha\beta} \quad , \quad g^{\alpha\beta} = A^{\alpha\beta} \equiv \delta^{\alpha\beta} \ . \tag{8.9}$$

The forms for $g_{\alpha\beta}$ used by REPSIL and PETROS 2 are identical and are based on the thin shell assumption. On the other hand REPSIL employs the exact inverse $g^{\alpha\beta}$ to $g_{\alpha\beta}$, while PETROS 2 uses an approximate thin shell inverse. PETROS 1 implicitly assumes that the metric and inverse are equal to the initial middle surface metric and inverse, which for the particular initial geometry treated (the cylinder section) equal the Kronecker delta. Hence, PETROS 1 becomes more restrictive than either REPSIL and PETROS 2 by assuming the classical thin shell approximation and neglecting any variation through the thickness (cf. $(7.8)_1$). Moreover, the distortion of the middle surface is neglected. Thus, PETROS 1 uses an approximation to the thin shell expressions used by REPSIL and PETROS 2 for the metric $g_{\alpha\beta}$ and inverse $g^{\alpha\beta}$, which is valid for extremely thin shells that suffer negligible middle surface distortions.

REPSIL and PETROS 2 employ mixed components in their expressions for determining the stress, while the PETROS 1 expressions do not distinguish covariant from contravariant components due to the special form of its metric. However, for the sake of comparison we shall also write the PETROS 1 expression in mixed components, summarizing the expressions used by these codes for determining the stress as follows:

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REPSIL (cf. (7.28) - (7.31), (7.33))

$$\overset{T}{\sigma}{}^{\alpha}_{\beta} = \sigma^{-\alpha}_{\ \beta} + \frac{E}{1+\nu}\,(\Delta E^{\alpha}_{\beta} + \frac{\nu}{1-\nu}\,\Delta E^{\gamma}_{\gamma}\,\delta^{\alpha}_{\beta})\ ,$$

$$\overset{C}{\sigma}{}^{\alpha}_{\beta} = \sigma^{-\alpha}_{\ \beta} - \frac{1-2\nu}{3(1-\nu)}\,\sigma^{-\gamma}_{\ \gamma}\,\delta^{\alpha}_{\beta}\ , \tag{8.10}$$

$$\sigma^{\alpha}_{\beta} = \overset{T}{\sigma}{}^{\alpha}_{\beta} - \Delta\lambda\,\overset{C}{\sigma}{}^{\alpha}_{\beta}\ , \qquad A\,\Delta\lambda^2 - 2B\,\Delta\lambda + C = 0\ ,$$

PETROS 2 [2; Eq. 5.68 - 5.75]

$$\overset{T}{\sigma}{}^{\alpha}_{\beta} = \sigma^{-\alpha}_{\ \beta} + \frac{E}{1+\nu}\,(\Delta E^{\alpha}_{\beta} + \frac{\nu}{1-\nu}\,\Delta E^{\gamma}_{\gamma}\,\delta^{\alpha}_{\beta})\ ,$$

$$\overset{C}{\sigma}{}^{\alpha}_{\beta} = \overset{T}{\sigma}{}^{\alpha}_{\beta} - \frac{1-2\nu}{3(1-\nu)}\,\overset{T}{\sigma}{}^{\gamma}_{\gamma}\,\delta^{\gamma}_{\beta}\ , \tag{8.11}$$

$$\sigma^{\alpha}_{\beta} = \overset{T}{\sigma}{}^{\alpha}_{\beta} - \Delta\lambda\,\overset{C}{\sigma}{}^{\alpha}_{\beta}\ , \qquad A\,\Delta\lambda^2 - 2B\,\Delta\lambda + C = 0\ ,$$

PETROS 1 [1; Sect. 4.3]

$$\overset{T}{\sigma}{}^{\alpha}_{\beta} = \sigma^{-\alpha}_{\ \beta} + \frac{E}{1+\nu}\,(\Delta E^{\alpha}_{\beta} + \frac{\nu}{1-\nu}\,\Delta E^{\gamma}_{\gamma}\,\delta^{\alpha}_{\beta})\ ,$$

$$\overset{C}{\sigma}{}^{\alpha}_{\beta} = \overset{T}{\sigma}{}^{\alpha}_{\beta} - \frac{1-2\nu}{3(1-\nu)}\,\overset{T}{\sigma}{}^{\gamma}_{\gamma}\,\delta^{\alpha}_{\beta}\ , \tag{8.12}$$

$$\sigma^{\alpha}_{\beta} = \overset{T}{\sigma}{}^{\alpha}_{\beta} - \Delta\lambda\,\overset{C}{\sigma}{}^{\alpha}_{\beta}\ , \qquad - 2B\,\Delta\lambda + C = 0\ ,$$

where mixed components are given by (7.27) & (7.34) and the coefficient
A, B, C are defined uniformly as

$$A = \overset{C}{\sigma}{}^{\alpha}_{\beta}\,\overset{C}{\sigma}{}^{\beta}_{\alpha} - \frac{1}{3}\,\overset{C}{\sigma}{}^{\alpha}_{\alpha}\,\overset{C}{\sigma}{}^{\beta}_{\beta}\ ,$$

$$B = \overset{C}{\sigma}{}^{\alpha}_{\beta}\,\overset{T}{\sigma}{}^{\beta}_{\alpha} - \frac{1}{3}\,\overset{C}{\sigma}{}^{\alpha}_{\alpha}\,\overset{T}{\sigma}{}^{\beta}_{\beta}\ , \tag{8.13}$$

$$C = \overset{T}{\sigma}{}^{\alpha}_{\beta}\,\overset{T}{\sigma}{}^{\beta}_{\alpha} - \frac{1}{3}\,\overset{T}{\sigma}{}^{\alpha}_{\alpha}\,\overset{T}{\sigma}{}^{\beta}_{\beta} - \frac{2}{3}\,\sigma^2_o\ ,$$

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in all the above formulae. The first difference to be noted in these formulations is that while REPSIL and PETROS 2 determine $\Delta\lambda$ from the exact quadratic formulae thus assuring a stress σ^{α}_{β} on the yield surface, PETROS 1 uses a linear approximation to determine $\Delta\lambda$ giving a stress not necessarily on the yield surface. HUFFINGTON [4; Sect. V] has assessed the possible errors resulting from the use of this linear approximation, indicating that in a typical problem involving the large transient deformation of a cylindrical panel on the average a discrepancy of 50% in the stresses can occur which initially can go as high as 210%. Hence, we can conclude that the use of a linear approximation for $\Delta\lambda$ in this type of problem is certainly inappropriate.

A second not so obvious difference involves the stress components used to determine corrector stress components $\overset{C}{\sigma^{\alpha}_{\beta}}$, PETROS 1 and PETROS 2 using $\overset{T}{\sigma^{\alpha}_{\beta}}$ instead of $\sigma^{-\alpha}_{\beta}$ as REPSIL does. The use of $\overset{T}{\sigma^{\alpha}_{\beta}}$ is rationalized in [2; Sect. 5.4.3] by the observation that a real value for $\Delta\lambda$ is more likely to result by its use. However, this ad hoc reasoning becomes much less persuasive when it is realized that the plastic flow rule (C.10) is violated by this choice, because $\overset{T}{\sigma^{\alpha}_{\beta}}$ lies outside rather than on the yield surface. When we add to this the fact that the procedure using $\sigma^{-\alpha}_{\beta}$ outlined in [4; Sect. IV] and used in REPSIL has proved in practice completely adequate, giving real $\Delta\lambda$ for all reasonable deformations, the choice of $\overset{T}{\sigma^{\alpha}_{\beta}}$ become still less justified.

Lastly, we point out that all three codes have the capability of treating strain rate sensitive and strain hardening materials, and do this in more or less the same manner using a mechanical sublayer model for strain hardening and an exponential strain rate dependent yield stress [2; Sect. 5.4.2].

8.1.3 Equation of Motion and Associated Resultants Formulation.
The various forms of the equations of motions used by these codes are limited by the common assumptions that

$$\underset{\sim}{c} = 0 \quad , \quad \gamma_1 = 0 \quad , \quad \gamma_2 = 0 \ . \qquad (8.14)$$

Hence, comparison of these equations will be within the limits imposed by these assumptions. The equations of motion and the associated membrane and bending resultant relations used in these codes can be summarized as follows:

REPSIL (cf. (7.14) - (7.16))

$$\gamma^*_0 \ \dot{\underset{\sim}{v}} = (\hat{M}^{*\beta\alpha} \ \underset{\sim}{n})_{,\beta\alpha} + \hat{\underset{\sim}{N}}^{*\alpha}{}_{,\alpha} + \underset{\sim}{F}^* \ ,$$

$$\gamma^*_0 = \rho_0 \ h \ A^{\frac{1}{2}} \quad , \qquad \hat{\underset{\sim}{N}}^* = \hat{Q}^{*\alpha\beta} \ \underset{\sim}{a}_\beta + \Gamma_{\beta\gamma}{}^\alpha \ \hat{M}^{*\beta\gamma} \ \underset{\sim}{n} \ ,$$

$$\hat{Q}^{*\alpha\beta} = a^{\frac{1}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{\alpha\beta} \ (1 - \zeta \ b^\gamma_\gamma) \ d\zeta \ , \qquad (8.15)$$

$$\hat{M}^{*\alpha\beta} = a^{\frac{1}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} [\sigma^{\alpha\beta}(1 - \zeta \ b^\gamma_\gamma) - \zeta \ b^{(\alpha}_\gamma \ \sigma^{\gamma\beta)}] \ \zeta \ d\zeta \ ,$$

PETROS 2 [2; Eq. 2.67, 2.69, 2.83, 2.87, 2.88]

$$\gamma^*_0 \ \dot{\underset{\sim}{v}} = (M^{*\beta\alpha}{}_{,\beta} \ \underset{\sim}{n})_{,\alpha} + \underset{\sim}{N}^{*\alpha}{}_{,\alpha} + \underset{\sim}{F}^* \ ,$$

$$\gamma^*_0 = \rho_0 \ h \ A^{\frac{1}{2}} \quad , \qquad \underset{\sim}{N}^* = Q^{*\alpha\beta} \ \underset{\sim}{a}_\beta + \Gamma_{\beta\gamma}{}^\alpha \ M^{*\beta\gamma} \ \underset{\sim}{n} \ ,$$

$$Q^{*\alpha\beta} = a^{\frac{1}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} (\delta^\beta_\gamma - \zeta \ b^\beta_\gamma) \ (1 - \zeta \ b^\delta_\delta) \ \sigma^{\gamma\alpha} \ d\zeta \ , \qquad (8.16)$$

$$M^{*\alpha\beta} = a^{\frac{1}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} (\delta^\beta_\gamma - \zeta \ b^\beta_\gamma) \ (1 - \zeta \ b^\delta_\delta) \ \sigma^{\gamma\alpha} \ \zeta \ d\zeta \ ,$$

the current Christoffel symbols can be used in the PETROS 1 formulation, the right hand side of $(8.17)_1$ makes no restriction as to the smallness of the membrane deformation. Finally, comparison of $(8.17)_{4,5}$ with $(7.8)_{5,7}$ relations $(8.17)_{4,5}$ with $(7.8)_{5,7}$ shows that the PETROS 1 resultant relations assume the classical thin shell approximation, neglecting terms of order $|\zeta\, b^\alpha_\beta|$.

8.1.4 <u>Summary of Comparison and Conclusions</u>. The difference between the formulations used in these codes are summarized in Table 8.1. From it we are able to make a number of useful conclusions about the kind of problems in which these codes can be expected to give good results. First, we emphasize that all the formulations neglect inertia terms containing γ_1 and γ_2, so that as pointed out in Section 7.2 the accuracy of solutions will diminish whenever harmonics of the same or shorter wavelength than the thickness are present. REPSIL and PETROS 2 are virtually based on equivalent formulations, both being able to treat the finite deformation of thin shells for which the Kirchhoff hypothesis is a good approximation. Only in the characterization of the flow rule do they markedly differ, the REPSIL formulation being more in line with classical plasticity, while the PETROS 2 formulation approaches the classical case only as the strain increments become smaller.

PETROS 1 is not as general as the other codes, which is to be expected since it is the progenitor of the other two. The linearized strain-displacement relation restricts it to small displacement increments for which rotations are small. Also, the improper characterization of the curvature effects in this relation make the formulation appropriate only for virtually flat shells. Due to the formulation of the elastoplastic stress calculation and to a smaller extent the equation of motion, the membrane deformations must be small. In addition the flow rule characterization and the linearized $\Delta\lambda$ calculation require for accuracy that the strain increment be small. Hence, only for problems involving the deformation of thin shells that

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Table 8.1 Differences in the Theoretical Formulations Used in the REPSIL, PETROS 2 and PETROS 1 Computer Codes

Formulation	Property	REPSIL	PETROS 2	PETROS 1
Strain Displacement Relation	Displacement Increment	Exact	Exact	Linearized
	Shell Curvature	Thin Shell; Neglects Terms $O\lvert\zeta\, b_\beta^\alpha\rvert^2$	Thin Shell; Neglects Terms $O\lvert\zeta\, b_\beta^\alpha\rvert^2$	Flat Thin Shell; Neglects Terms $O\lvert\zeta\, b_\beta^\alpha\rvert^2$
	Membrane Deformation	Exact	Exact	Neglected
Elastoplastic Stress Calculation	Shell Curvature	Thin Shell; Neglects Terms $O\lvert\zeta\, b_\beta^\alpha\rvert^2$	Thin Shell; Neglects Terms $O\lvert\zeta\, b_\beta^\alpha\rvert^2$	Classical Thin Shell; Neglects Terms $O\lvert\zeta\, b_\beta^\alpha\rvert^2$
	Δλ Calculation	Exact (quadratic)	Exact (quadratic)	Linearized
	Flow Rule Corrector Stress	On Yield Surface	Outside Yield Surface	Outside Yield Surface
Resultant Relations	Shell Curvature	Thin Shell; Neglects Terms $O\lvert\zeta\, b_\beta^\alpha\rvert^2$	Thin Shell; Neglects Terms $O\lvert\zeta\, b_\beta^\alpha\rvert^2$	Classical Thin Shell; Neglects Terms $O\lvert\zeta\, b_\beta^\alpha\rvert^2$
Equation of Motion	Consv. of Mass	Exact	Exact	Neglects Membrane Deformation
	Christoffel Symbols	Exact	Exact	Exact

remain virtually flat and undergo little membrane deformation and small incremental displacements can the PETROS 1 code be expected on theoretical grounds to have the accuracy of REPSIL or PETROS 2.

An assessment of the relative merit of PETROS 1 and PETROS 2 in predicting the known results of a physical experiment is attempted in [2; Sect. 6.3.2 & 6.3.3]. However, due to the apparent lack of precise control of the edge conditions during the experiment and the use of a rather small number of mesh points in the codes, the superiority of PETROS 2 over PETROS 1 was not definitively proven.

8.2 Comparison of Major Computational Differences

Similarities in their theoretical formulations make these codes broadly similar computationally. All solve finite difference representations of the particular partial differential equations forming the bases of each code. They all attain solutions by repeating a sequence of calculations which advance the values of variables one time step, similar to the sequence described in Section 7.5. On the other hand, differences in the theoretical equations forming their bases, as described in Section 8.1, have resulted in differences in the way these codes are computationally formulated. Differences also arise from dissimilar finite difference representations of the same formulae. Moreover, there are differences due to different computational capabilities. We shall not go into all the computational differences between these codes, but only describe the more salient ones.

The various forms of the equation of motion used by these codes give rise to major differences in the finite difference representations of the second partials of the bending components in these equations. From $(8.16)_1$ and $(8.17)_1$ it is clear that in PETROS 1 and PETROS 2 the bending components occur in essentially the same combination, viz. $(M^{\beta\alpha}{}_{,\beta}\, n)_{,\alpha}$, while $(8.15)_1$ gives the combination $(\hat{M}^{\beta\alpha}\, n)_{,\beta\alpha}$ for REPSIL. For a typical diagonal component, say M^{11}, PETROS 1 uses at mesh point (i,j) the obvious finite difference representation

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$$\left[(M^{11},_1\ \underset{\sim}{n}),_1 \right]_{(i,j)} = \frac{1}{4(\Delta\xi^1)^2} \left[M^{11}(i+2,j) - M^{11}(i,j) \right) \underset{\sim}{n}(i+1,j)$$

$$\tag{8.19}$$
$$- \left(M^{11}(i,j) - M^{11}(i-2,j) \right) \underset{\sim}{n}(i-1,j) \right] \ .$$

This formula suffers from the disadvantage that the domain of mesh points for the values of M^{11} skips the values at the points neighboring (i,j) and extends over an interval spanning five points. In other words, the domain of M^{11} is too coarse and too extended. PETROS 2 circumvents this situation by the judicious use of forward and backward difference operators:

$$\left[(M^{11},_1\ \underset{\sim}{n}),_1 \right]_{(i,j)} = \frac{1}{4(\Delta\xi^1)^2} \left[(3M^{11}(i+1,j) - 4M^{11}(i,j) + M^{11}(i-1,j) \right) \underset{\sim}{n}(i+1,j)$$

$$\tag{8.20}$$
$$+ \left(3M^{11}(i-1,j) - 4M^{11}(i,j) + M^{11}(i+1,j) \right) \underset{\sim}{n}(i-1,j) \right] \ ,$$

resulting in a refined and compact domain, in which only the values at (i,j) and its two nearest neighbors are required. Due to the different form of the equation of motion used in REPSIL, this problem is not even encountered, for the straight forward use of the finite difference operator for second derivatives gives

$$\left[(\hat{M}^{11}\ \underset{\sim}{n}),_{11} \right]_{(i,j)} = \frac{1}{(\Delta\xi^1)^2} \left[\hat{M}^{11}(i+1,j)\ \underset{\sim}{n}(i+1,j) \right.$$

$$\tag{8.21}$$
$$\left. - 2\hat{M}^{11}(i,j)\ \underset{\sim}{n}(i,j) + \hat{M}^{11}(i-1,j)\ \underset{\sim}{n}(i-1,j) \right] \ .$$

Not only does this formulation result in a refined and compact domain for diagonal bending components, but also a refined domain for $\underset{\sim}{n}$ by including its value at (i,j).

The various definitions of the membrane and bending components lead to differences in the storage requirements of arrays. From $(8.17)_{4,5}$ it is clear that as used in PETROS 1 $Q^{\alpha\beta}$ and $M^{\alpha\beta}$ are symmetric 2×2 matrices. However, as shown in the previous section, the effects of the middle surface curvature have been ignored through

the neglect of $\zeta\, b_\beta^\alpha$ terms. When this curvature is taken into consideration, the direct generalization of $Q^{\alpha\beta}$ and $M^{\alpha\beta}$ results in definitions similar to those of PETROS 2, as given by $(8.16)_{4,5}$. Clearly the symmetry property is lost; $Q*^{\alpha\beta}$ and $M*^{\alpha\beta}$ are nonsymmetric 2×2 matrices. Hence the number of independent elements has been raised from six to eight. Since the equations of motion necessitate the finite differencing of these components, their values are most conveniently stored as arrays. This means that in generalizing from PETROS 1 to PETROS 2 it has become necessary to store two additional arrays. On the other hand, the generalization used in REPSIL to account for curvature effects retains a symmetric formulation with $\hat{Q}*^{\alpha\beta}$ and $\hat{M}*^{\alpha\beta}$ as defined by $(8.15)_{4,5}$, both symmetric 2×2 matrices. Hence the REPSIL generalization of the PETROS 1 formulation does not necessitate the storage of two additional arrays for the membrane and bending components.

Also the REPSIL and PETROS 2 formulations use the starred variables $\hat{Q}*^{\alpha\beta}$, $\hat{M}*^{\alpha\beta}$ and $Q*^{\alpha\beta}$, $M*^{\alpha\beta}$ rather than the variables $Q^{\alpha\beta}$, $M^{\alpha\beta}$ as does PETROS 1. This results in the reduced use of the Christoffel symbol in the equations of motion, since partial rather than covariant derivatives predominate, making REPSIL and PETROS 2 computationally simpler than PETROS 1.

As already mentioned, these codes treat similar analytic relations in computationally different ways. The numerical integration of the stress components and their moments through the shell thickness for the membrane and bending components is accomplished differently. REPSIL takes straightforward sums of the values of the stress and its moment at the middle-point of each layer, as does PETROS 1. PETROS 2 uses Gaussian quadrature for this integration. In [2; Sect. 6.2.2] the superiority of Gaussian quadrature over middle-point integration is demonstrated in a small deflection elastic problem for which Gaussian quadrature models the deformation through the thickness exactly. However, it does not necessarily follow that this superiority would be so pronounced in a large deflection problem, wherein gross plastic deformation is present.

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Also, these codes use different finite difference operators for the same boundary conditions. PETROS 1 approximates the two boundary conditions it treats, the symmetry edge and clamped edge, by central difference operators of order $|\Delta\xi^\alpha|^2$. However, as pointed out in [3], the clamped edge condition as formulated in the code itself is improper. Early versions of REPSIL employed the same boundary conditions using the same operators, but with the clamped edge condition correctly formulated. The present version of REPSIL retains the central difference operator for the symmetry edge, but uses forward and backward difference operators of order $|\Delta\xi^\alpha|^2$ for the clamped edge. In addition REPSIL treats the hinged edge using forward and backward operators of order $|\Delta\xi^\alpha|^2$. PETROS 2 treats the symmetry edge, clamped edge and hinged edge conditions more or less in the same way as REPSIL. It also has the capability of dealing with the free edge and cyclic edge conditions, using central difference operators for both [2; Sect. 4.3]. Except for the free edge difference operator which is of order $|\Delta\xi^\alpha|$, all the operators used in PETROS 2 are of order $|\Delta\xi^\alpha|^2$. As of now it is not clear whether central or forward and backward operators are preferable and more work needs to be done here.

The comparison of the difference operators used at the boundaries points out some differences in the computational capabilities of these codes. These codes also have different capabilities as to accommodating various initial shell geometries. As already noted, PETROS 1 can only treat initially cylindrical panels. REPSIL and PETROS 2 have been formulated so as to accept any initial geometry, provided the middle surface is simply connected and bounded by four smooth edges. This versatility is achieved by confining the specification of the initial shape of the middle surface to one simple subprogram, thus making the remainder of the codes independent of initial geometry. At present REPSIL has initial geometry subprograms for a flat plate, cylinder and cylindrical ogive. PETROS 2 has subprograms for a flat plate, cylinder and cone. Also through the manipulation of values at the boundaries to eliminate the dependence on one of shell parameters

Table 8.2 Computational Differences and Capabilities of the
REPSIL, PETROS 2 and PETROS 1 Computer Codes

		REPSIL	PETROS 2	PETROS 1
Domain of Bending Components		Refined & Compact	Refined & Compact	Coarse & Extended
Membrane & Bending Components		Symmetric Matrices	Nonsymmetric Matrices	Symmetric Matrices
Christoffel Symbol Usage		Minimal	Minimal	Extensive
Thickness Integration		Middle-Point	Gaussian	Middle-Point
Initial Geometry		4 Smooth Sided Surface	4 Smooth Sided Surface	Cylinder Section
Loading Options	Initial Velocity	Yes	Yes	Yes
	Pressure	Yes	Yes	No
Boundary Conditions	Symmetry Edge	Central Diff $O\lvert\Delta\xi\rvert^2$	Central Diff $O\lvert\Delta\xi\rvert^2$	Central Diff $O\lvert\Delta\xi\rvert^2$
	Clamped Edge	F & B Diff $O\lvert\Delta\xi\rvert^2$	F & B Diff $O\lvert\Delta\xi\rvert^2$	Central Diff $O\lvert\Delta\xi\rvert^2$
	Hinged Edge	F & B Diff $O\lvert\Delta\xi\rvert^2$	F & B Diff $O\lvert\Delta\xi\rvert^2$	N/A
	Free Edge	N/A	Central Diff $O\lvert\Delta\xi\rvert$	N/A
	Cyclic Edge	N/A	Central Diff $O\lvert\Delta\xi\rvert^2$	N/A

ξ^{α}, PETROS 2 can treat certain symmetric deformations of similarly symmetric shells; namely, axisymmetric deformations and plane strain deformations [2; Sect. 4.3.7 & 4.3.8].

Both REPSIL and PETROS 2 are capable of handling more general load situations than PETROS 1. PETROS 1 can only handle an initial impulse velocity loading. REPSIL and PETROS 2 can also accept a time and space varying pressure loading. This loading can be specified either analytically or numerically.

The results of this section are conveniently summarized in Table 8.1. It shows that computationally REPSIL and PETROS 2 are more or less equal and both are more sophisticated than PETROS 1. REPSIL has a slight advantage over PETROS 2 in the use of symmetric membrane and bending components, while PETROS 2 uses the superior Gaussian quadrature to integrate through the thickness. As far as capabilities are concerned, PETROS 2 has the most options available and REPSIL has more than PETROS 1. As is to be expected, REPSIL and PETROS 2 are comparable and both are superior to their older predecessor PETROS 1.

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APPENDIX A. SURFACE INTEGRAL THEOREMS

We derive in this appendix the surface integral counterparts of two volume integral theorems used in continuum mechanics. First we derive Reynold's transport theorem, which characterizes how the operations of time differentiation and space integration commute for a time dependent surface. Let $S(t)$ be a time dependent surface in 3-space generated as the image of the region Ω in 2-space under the mapping

$$\underset{\sim}{r} = \underset{\sim}{r} \, (\xi^{\alpha}, t) \quad ; \quad \xi^{\alpha} \, \epsilon \, \Omega \tag{A.1}$$

and let $\oint (\xi^{\alpha}, t)$ be a time dependent function defined on Ω. The integral of $\oint$ over $S(t)$ can be written as

$$\iint\limits_{S(t)} \oint \, da = \iint\limits_{\Omega} \oint \, a^{\frac{1}{2}} \, d\xi^1 \, d\xi^2 \quad ,^* \tag{A.2}$$

where a is the determinant (2.7). By virtue of (2.28), the time derivative of (A.2) becomes

$$\frac{d}{dt} \iint\limits_{S(t)} \oint \, da = \iint\limits_{\Omega} (\dot{\oint} + \nabla \cdot \underset{\sim}{v} \, \oint) \, a^{\frac{1}{2}} \, d\xi^1 \, d\xi^2 \quad . \tag{A.3}$$

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* See BUCK [16, Eq. 7-3].

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Applying (A.2) in reverse to (A.3), we obtain *Reynold's transport theorem* for surfaces:

$$\frac{d}{dt} \iint\limits_{S(t)} \not{b} \; da = \iint\limits_{S(t)} (\dot{\not{b}} + \nabla \cdot \underset{\sim}{v} \not{b}) \; da \quad .$$ (A.4)

Next we obtain the divergence theorem, which converts surface integrals into line integrals along the surface boundaries, by specializing Stoke's theorem[*] to surface vectors. For a continuous vector function $\not{b}$ defined on a smooth surface S bounded by the piecewise smooth curve C, Stoke's theorem reads

$$\iint\limits_{S} \nabla \times \underset{\sim}{\not{b}} \cdot \underset{\sim}{n} \; da = \oint\limits_{C} \underset{\sim}{\not{b}} \cdot \underset{\sim}{\tau} \; ds \quad ,$$ (A.5)

with $\underset{\sim}{\tau}$ the unit tangent to the curve C. The positive sense of $\underset{\sim}{\tau}$ along C is determined as usual by the right-hand-screw relative to the surface normal $\underset{\sim}{n}$ (see Figure 5.1). The following relations will hold between $\underset{\sim}{n}$, $\underset{\sim}{\tau}$ and the exterior unit normal $\underset{\sim}{v}$ to C:

$$\underset{\sim}{\tau} = \underset{\sim}{n} \times \underset{\sim}{v} \quad , \qquad \underset{\sim}{v} = \underset{\sim}{\tau} \times \underset{\sim}{n} \quad , \qquad \underset{\sim}{n} = \underset{\sim}{v} \times \underset{\sim}{\tau} \quad .$$ (A.6)

Since $\underset{\sim}{\tau}$ and $\underset{\sim}{v}$ are normal to $\underset{\sim}{n}$, $\underset{\sim}{a}_\alpha$ and $\underset{\sim}{a}^\alpha$ can be used as bases to resolve these vectors

$$\underset{\sim}{\tau} = \tau^\alpha \underset{\sim}{a}_\alpha = \tau_\alpha \underset{\sim}{a}^\alpha \quad , \qquad \underset{\sim}{v} = v^\alpha \underset{\sim}{a}_\alpha = v_\alpha \underset{\sim}{a}^\alpha$$ (A.7)

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[*]For a proof of Stoke's theorem see, for example, KELLOGG [17; p. 89].

and conversely $\underset{\sim}{\tau}, \underset{\sim}{\nu}$ can be used as a basis for $\underset{\sim}{a}_\alpha$ and $\underset{\sim}{a}^\alpha$:

$$\underset{\sim}{a}_\alpha = \tau_\alpha \underset{\sim}{\tau} + \nu_\alpha \underset{\sim}{\nu} \qquad , \qquad \underset{\sim}{a}^\alpha = \tau^\alpha \underset{\sim}{\tau} + \nu^\alpha \underset{\sim}{\tau} \quad . \qquad \text{(A.8)}$$

Moreover, the components of $\underset{\sim}{\tau}$ and $\underset{\sim}{\nu}$ satisfy the relations

$$\tau^\alpha = \epsilon^{\beta\alpha} \nu_\beta \qquad , \qquad \nu_\alpha = \epsilon_{\alpha\beta} \tau^\beta \quad . \qquad \text{(A.9)}$$

If we let $\underset{\sim}{b}$ be a vector tangent to the surface S, so that

$$\underset{\sim}{b} = b_\beta \underset{\sim}{a}^\beta \qquad ,$$

we find that

$$\nabla \times \underset{\sim}{\dot{b}} \cdot \underset{\sim}{n} = \underset{\sim}{n} \cdot \underset{\sim}{a}^\alpha \times (b_\beta \underset{\sim}{a}^\beta)_{,\alpha} = \epsilon^{\alpha\beta} b_{\beta|\alpha} \quad .$$

Stoke's theorem then becomes

$$\iint_S \epsilon^{\alpha\beta} b_{\beta|\alpha} \, da = \oint_C b_\beta \tau^\beta \, ds \quad . \qquad \text{(A.10)}$$

Finally, setting

$$g^\alpha = \epsilon^{\alpha\beta} b_\beta$$

and substituting in (A.10), we obtain, in view of (A.9), the *divergence theorem* for surfaces:

$$\iint_S g^\alpha{}_{|\alpha} \, da = \oint_C g^\alpha \nu_\alpha \, ds \quad . \qquad \text{(A.11)}$$

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APPENDIX B. SYMMETRY BOUNDARY FORMULATION

A *symmetry plane* is a plane about which the shell is always sym-
metrically positioned. Hence initially the shell is symmetrically
distributed with respect to the symmetry plane and subsequently deforms
in a symmetric manner about this plane. Consequently, the body force b
and surface tractions t are required to be symmetrically distributed
with respect to the symmetry plane. The intersection of the middle
surface with the symmetry plane gives the *symmetry edge* or *boundary*.
For convenience, we take the symmetry boundary to be a coordinate curve
and require the remaining coordinate curves to be symmetrically dis-
tributed relative to the symmetry plane.

We first let $\xi^1 = \xi^1_o$ be the symmetry boundary. The symmetry
plane is located a perpendicular distance K_1 from the origin and oriented
relative to the fixed orthonormal basis k_i ($i = 1,2,3$) such that k_1 is
normal and hence k_σ ($\sigma = 2,3$) parallel, as shown in Figure 6.1.

The position vector to any point on the symmetry edge by definition
satisfies

$$r\,(\xi^1_o,\ \xi^2,\ t) \cdot k_1 = K_1 \quad . \tag{B.1}$$

Also, using the symplifying notation

$$\not{b}_+ = \not{b}\,(\xi^1_o + \Delta\xi^1,\ \xi^2),\quad \not{b}_- = \not{b}\,(\xi^1_o - \Delta\xi^1,\ \xi^2),\ \not{b} = \not{b}\,(\xi^1_o,\ \xi^2)\ ,$$

we obtain as a consequence of the symmetric distribution of coordinate
curves about the symmetry plane

$$[r_+ + r_-] \cdot k_1 = 2\,r \cdot k_1 = 2K_1,\quad [r_+ - r_-] \cdot k_\sigma = 0 \quad (\sigma = 2,3). \tag{B.2}$$

Agreeing to designate components with respect to the basis k_i by sub-scripts inclosed in parentheses, the last relations become

$$r_{(1)+} + r_{(1)-} = 2 K_1 \quad , \quad r_{(\sigma)+} = r_{(\sigma)-} \quad (\sigma = 2,3) \quad . \quad (B.3)$$

The symmetric distribution of the body force b and surface tractions t is equivalent to the conditions

$$b_{(1)+} = - b_{(1)-} \quad , \qquad b_{(\sigma)+} = b_{(\sigma)-} \quad ,$$

$$t_{(1)+} = - t_{(1)-} \quad , \qquad t_{(\sigma)+} = t_{(\sigma)-} \quad . \qquad (B.4)$$

The symmetry properties of all other variables follow from (B.3) and (B.4). In particular, from (4.33) and (4.34) we have

$$F_{(1)+} = - F_{(1)-} \quad , \qquad F_{(\sigma)+} = F_{(\sigma)-} \quad ,$$

$$C_{(1)+} = - C_{(1)-} \quad , \qquad C_{(\sigma)+} = C_{(\sigma)-} \quad . \qquad (B.5)$$

Before continuing with determining the symmetry properties of the remaining variables, we must first indicate how the symmetry properties of variables change under differentiation. The above relations clearly show that the components of vectors with respect to the basis k_i are of two types: symmetric and antisymmetric, satisfying respectively the typical relations

$$\phi_+ = \phi_- \quad , \qquad \phi_+ = - \phi_- \quad .$$

It follows from the definition of the derivative that differentiation with respect to ξ^1 will change the symmetry type of a variable while differentiation with respect to ξ^2 will not, so that

$$b_+ = b_- \quad \Rightarrow \quad b_{,1+} = - b_{,1-} \quad , \quad b_{,2+} = b_{,2-} \quad ,$$

$$b_+ = - b_- \quad \Rightarrow \quad b_{,1+} = b_{,1-} \quad , \quad b_{,2+} = - b_{,2-} \quad .$$

With these properties, we obtain on differentiating (B.3) the symmetry properties of the basis $\underline{a}_\alpha$:

$$a_{1(1)+} = a_{1(1)-} \quad , \qquad a_{1(\sigma)+} = - a_{1(\sigma)-} \quad ,$$

$$\tag{B.6}$$

$$a_{2(1)+} = - a_{2(1)-} \quad , \qquad a_{2(\sigma)+} = a_{2(\sigma)-} \quad ,$$

from which the symmetry properties of the metric $a_{\alpha\beta}$ follow

$$a_{11+} = a_{11-} \quad , \quad a_{12+} = - a_{12-} \quad , \quad a_{22+} = a_{22-} \quad . \tag{B.7}$$

Using (2.4) and (2.10) we find the symmetry properties of $\underline{n}$ and $\underline{a}^\alpha$:

$$n_{(1)+} = - n_{(1)-} \quad , \qquad n_{(\sigma)+} = n_{(\sigma)-} \quad ,$$

$$a^1_{(1)+} = a^1_{(1)-} \quad , \qquad a^1_{(\sigma)+} = - a^1_{(\sigma)-} \quad , \tag{B.8}$$

$$a^2_{(1)+} = - a^2_{(1)-} \quad , \qquad a^2_{(\sigma)+} = a^2_{(\sigma)-} \quad .$$

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Differentiation of (B.6) gives us

$$a_{1,1(1)+} = - a_{1,1(1)-} \quad , \quad a_{1,1(\sigma)+} = a_{1,1(\sigma)-} \quad ,$$

$$a_{1,2(1)+} = a_{1,2(1)-} \quad , \quad a_{1,2(\sigma)+} = - a_{1,2(\sigma)-} \quad , \qquad (B.9)$$

$$a_{2,2(1)+} = - a_{2,2(1)-} \quad , \quad a_{2,2(\sigma)+} = a_{2,2(\sigma)-} \quad .$$

Combining the last two results, we obtain the symmetry properties of $b_{\alpha\beta}$ and $\Gamma_{\alpha\beta}{}^{\gamma}$:

$$b_{11+} = b_{11-} \quad , \quad b_{12+} = - b_{12-} \quad , \quad b_{22+} = b_{22-} \quad ,$$

$$\Gamma_{11+}^{1} = - \Gamma_{11-}^{1} \ , \ \Gamma_{12+}^{1} = \Gamma_{12-}^{1} \quad , \quad \Gamma_{22+}^{1} = - \Gamma_{22-}^{1} \ , \qquad (B.10)$$

$$\Gamma_{11+}^{2} = \Gamma_{11-}^{2} \quad , \quad \Gamma_{12+}^{2} = - \Gamma_{12-}^{2} \quad , \quad \Gamma_{22+}^{2} = \Gamma_{22-}^{2} \quad .$$

The definition (3.14) of the incremental strain $\Delta E_{\alpha\beta}$ shows that it has the same symmetry properties as $a_{\alpha\beta}$ and $b_{\alpha\beta}$

$$\Delta E_{11+} = \Delta E_{11-} \ , \quad \Delta E_{12+} = - \Delta E_{12-} \ , \quad \Delta E_{22+} = \Delta E_{22-} \quad , \qquad (B.11)$$

from which it follows that the stress $\sigma^{\alpha\beta}$ also has similar symmetry properties:

$$\sigma^{11}_{+} = \sigma^{11}_{-} \quad , \quad \sigma^{12}_{+} = - \sigma^{12}_{-} \quad , \quad \sigma^{22}_{+} = \sigma^{22}_{-} \quad , \qquad (B.12)$$

see Appendix C.

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The symmetry properties of the membrane and bending components we infer from their definitions (5.43), from which it follows that

$$\hat{Q}_+^{11} = \hat{Q}_-^{11} \quad , \quad \hat{Q}_+^{12} = - \hat{Q}_-^{12} \quad , \quad \hat{Q}_+^{22} = \hat{Q}_-^{22} \quad ,$$

$$\hat{M}_+^{11} = \hat{M}_-^{11} \quad , \quad \hat{M}_+^{12} = - \hat{M}_-^{12} \quad , \quad \hat{M}_+^{22} = \hat{M}_-^{22} \quad . \tag{B.13}$$

Also combining $(B.5)_{3,4}$ with $(B.8)_{3-5}$, we have

$$C_+^1 = - C_-^1 \quad , \quad C_+^2 = C_-^2 \tag{B.14}$$

Hence, by definitions (5.40), (5.44) and (5.46), we obtain

$$\hat{Q}_{*+}^{11} = \hat{Q}_{*-}^{11} \quad , \quad \hat{Q}_{*+}^{12} = - \hat{Q}_{*-}^{12} \quad , \quad \hat{Q}_{*+}^{22} = \hat{Q}_{*-}^{22} \quad ,$$

$$\hat{M}_{*+}^{11} = \hat{M}_{*-}^{11} \quad , \quad \hat{M}_{*+}^{12} = - \hat{M}_{*-}^{12} \quad , \quad \hat{M}_{*+}^{22} = \hat{M}_{*-}^{22} \quad , \tag{B.15}$$

$$\hat{N}_{*+}^1 = - \hat{N}_{*-}^1 \quad , \quad \hat{N}_{*+}^2 = \hat{N}_{*-}^2 \quad .$$

The symmetry properties of the velocities $\underline{v}$ and $\underline{\omega}$ easily follow from taking the time derivatives of (B.3) and $(B.8)_{1,2}$:

$$v_{(1)+} = - v_{(1)-} \quad , \quad v_{(\sigma)+} = v_{(\sigma)-} \quad ,$$

$$\omega_{(1)+} = - \omega_{(1)-} \quad , \quad \omega_{(\sigma)+} = \omega_{(\sigma)-} \quad . \tag{B.16}$$

To facilitate obtaining the symmetry conditions when the symmetry boundary is the $\xi^2 = \xi_0^2$ coordinate curve, we note that the symmetry characteristics of all the above variables are related to the number of times the index 1 appears: a variable is symmetric when the index 1 occurs an even number of time and antisymmetric when it occurs an odd number. This is a result not only of the symmetry edge being a $\xi^1 = \xi_0^1$ coordinate curve, but also of the symmetry plane being normal to basis vector $\underset{\sim}{k}_1$. Hence, for the case where $\xi^2 = \xi_0^2$ is the symmetry edge, the symmetry plane is positioned a perpendicular distance K_2 from the origin and oriented relative to the fixed basis $\underset{\sim}{k}_i$ such that $\underset{\sim}{k}_2$ is normal and hence $\underset{\sim}{k}_\tau$ ($\tau = 1,3$) parallel. By analogy with the previous case, we now have that variables in which the index 2 occurs an even number of times are symmetric and those in which it occurs an odd number antisymmetric. Thus, we have:

$$r_{(2)+} + r_{(2)-} = 2 K_2 \quad , \qquad r_{(\tau)+} = r_{(\tau)-} \quad , \qquad (\tau = 1,3) \qquad (B.17)$$

$$F_{(2)+} = - F_{(2)-} \quad , \qquad F_{(\tau)+} = F_{(\tau)-} \quad ,$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (B.18)$$

$$C_{(2)+} = - C_{(2)-} \quad , \qquad C_{(\tau)+} = C_{(\tau)-} \quad ,$$

$$a_{1(2)+} = - a_{1(2)-} \quad , \qquad a_{1(\tau)+} = a_{1(\tau)-} \quad ,$$

$$a_{2(2)+} = a_{2(2)-} \quad , \qquad a_{2(\tau)+} = - a_{2(\tau)-} \quad ,$$

$$a^1_{(2)+} = - a^1_{(2)-} \quad , \qquad a^1_{(\tau)+} = a^1_{(\tau)-} \quad , \qquad (B.19)$$

$$a^2_{(2)+} = a^2_{(2)-} \quad , \qquad a^2_{(\tau)+} = - a^2_{(\tau)-} \quad ,$$

$$n_{(2)+} = - n_{(2)-} \quad , \qquad n_{(\tau)+} = n_{(\tau)-} \quad ,$$

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$$a_{11+} = a_{11-} \quad , \quad a_{12+} = - a_{12-} \quad , \quad a_{22+} = a_{22-} \quad ,$$

$$b_{11+} = b_{11-} \quad , \quad b_{12+} = - b_{12-} \quad , \quad b_{22+} = b_{22-} \quad ,$$

$$\Gamma_{11+}^{1} = \Gamma_{11-}^{1} \quad \Gamma_{12+}^{1} = - \Gamma_{12-}^{1} \quad , \quad \Gamma_{22+}^{1} = \Gamma_{22-}^{1} \quad , \qquad (B.20)$$

$$\Gamma_{11+}^{2} = - \Gamma_{11-}^{2} \quad , \quad \Gamma_{12+}^{2} = \Gamma_{12-}^{2} \quad , \quad \Gamma_{22+}^{2} = - \Gamma_{22}^{2}$$

$$\Delta E_{11+} = \Delta E_{11-} \quad , \quad \Delta E_{12+} = - \Delta E_{12-} \quad , \quad \Delta E_{22+} = \Delta E_{22-} \quad ,$$

$$\qquad (B.21)$$

$$\sigma_{11+} = \sigma_{11-} \quad , \quad \sigma_{12+} = - \sigma_{12-} \quad , \quad \sigma_{22+} = \sigma_{22-} \quad ,$$

$$\hat{Q}_{\star+}^{11} = \hat{Q}_{\star-}^{11} \quad , \quad \hat{Q}_{\star+}^{12} = - \hat{Q}_{\star-}^{12} \quad , \quad \hat{Q}_{\star+}^{22} = \hat{Q}_{\star-}^{22} \quad ,$$

$$\hat{M}_{\star+}^{11} = \hat{M}_{\star-}^{11} \quad , \quad \hat{M}_{\star+}^{12} = - \hat{M}_{\star-}^{12} \quad , \quad \hat{M}_{\star+}^{22} = \hat{M}_{\star-}^{22} \quad , \qquad (B.22)$$

$$\hat{N}_{\star+}^{1} = \hat{N}_{\star-}^{1} \quad , \quad \hat{N}_{\star+}^{2} = - \hat{N}_{\star-}^{2} \quad ,$$

$$v_{(2)+} = - v_{(2)-} \quad , \quad v_{(\tau)+} = v_{(\tau)-} \quad ,$$

$$\qquad (B.23)$$

$$\omega_{(2)+} = - \omega_{(2)} \quad , \quad \omega_{(\tau)+} = \omega_{(\tau)-} \quad .$$

APPENDIX C. ELASTOPLASTIC STRESS FORMULATION

We present here a brief derivation of the formulation forming the basis of the stress calculation used in REPSIL. The derivation is similar to that given by HUFFINGTON [4; Sect.II], but generalized to general non-orthonormal bases. The material composing the shell is assumed to be isotropic. Also, the surface tractions are assumed so small compared to the internal stress field and the shell assumed so thin that the shear and normal stress components in the direction of the shell normal can be neglected. Hence, using components of the stress σ relative to the basis g_i, we have

$$\sigma^{31} = g^3 \cdot \sigma \cdot g^i = 0 \quad . \tag{C.1}$$

This gives us a state of plane stress in each lamella parallel to the middle surface. Using mixed components of stress, the nonzero components of stress are σ^α_β.[*]

The material is assumed to satisfy a linear incremental stress – strain relation in the elastic range. In the plastic range, the stress components are delimited by the von Mises-Hencky yield condition [18; p.20], which for the case of plane stress in mixed component form is

$$\phi(\sigma^\alpha_\beta) = \sigma^\alpha_\beta \sigma^\beta_\alpha - \frac{1}{3} (\sigma^\alpha_\alpha)^2 - \frac{2}{3} \sigma_o^2 \leqslant 0 \quad , \tag{C.2}$$

with σ_o the uniaxial yield stress.[**] Thus, the stress is restricted

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[*] Raising and lowering of indicies is accomplished through the use of metric $g_{\alpha\beta}$ and not $a_{\alpha\beta}$.

[**] This treatment applies to the mechanical sublayer model of strain hardening by considering the stress σ^α_β and yield stress σ_o to be those in an individual sublayer. Moreover, strain rate sensitivity is also included by making σ_o a function of the rate of strain. Cf. [2; Sect. 5.4.2].

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to values within the yield surface ($\phi < 0$) or on the yield surface ($\phi = 0$). Plastic straining only occurs when the stress has values on the yield surface.

We formulate an incremental theory of plastic behavior, assuming that the incremental strain components $\Delta E^{\alpha}_{\beta}$ are the sums of corresponding components of the elastic strain increment $\Delta \overset{E}{E}{}^{\alpha}_{\beta}$ and the plastic strain increment $\Delta \overset{P}{E}{}^{\alpha}_{\beta}$:

$$\Delta E^{\alpha}_{\beta} = \Delta \overset{E}{E}{}^{\alpha}_{\beta} + \Delta \overset{P}{E}{}^{\alpha}_{\beta} \quad . \tag{C.3}$$

The elastic strain increment components are related to the stress increment components through an isotropic Hooke's law:

$$\Delta \overset{E}{E}{}^{\alpha}_{\beta} = \frac{1+\nu}{E} \Delta \sigma^{\alpha}_{\beta} - \frac{\nu}{E} \Delta \sigma^{\gamma}_{\gamma} \delta^{\alpha}_{\beta} \quad . \tag{C.4}$$

Solving (C.4) for the stress components and using (C.3) we find

$$\Delta \sigma^{\alpha}_{\beta} = \frac{E}{1+\nu} [\Delta E^{\alpha}_{\beta} - \Delta \overset{P}{E}{}^{\alpha}_{\beta} + \frac{\nu}{1-\nu} \delta^{\alpha}_{\beta} (\Delta E^{\gamma}_{\gamma} - \Delta \overset{P}{E}{}^{\gamma}_{\gamma})] \quad . \tag{C.5}$$

Hence, if strain and hence stress increments occur in the time interval $[t - \Delta t, t]$, we have

$$\sigma^{\alpha}_{\beta}(t) = \sigma^{\alpha}_{\beta}(t - \Delta t) + \Delta \sigma^{\alpha}_{\beta} \quad . \tag{C.6}$$

Clearly, the state of stress at the time $t - \Delta t$ can be presumed to be an allowable state, so that

$$\phi(\sigma^{\alpha}_{\beta}(t - \Delta t)) \leqslant 0 \quad . \tag{C.7}$$

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Tentatively setting to zero the plastic strain increments in (C.5), if the state of stress at the time t is allowable, so that

$$\phi(\sigma_\beta^\alpha(t)) \leq 0 \quad , \tag{C.8}$$

then indeed the change in the stress state has occurred elastically and the stress at the time t is

$$\sigma_\beta^\alpha(t) = \sigma_\beta^\alpha(t - \Delta t) + \frac{E}{1+\nu} [\Delta E_\beta^\alpha + \frac{\nu}{1-\nu} \delta_\beta^\alpha \Delta E_\gamma^\gamma] \quad . \tag{C.9}$$

If however the stress at the time t is not allowable:

$$\phi(\sigma_\beta^\alpha(t)) > 0 \quad ,$$

then the plastic strain increments cannot be assumed to vanish, but must be determined from the flow rule. Using the yield condition (C.2) as a plastic potential (see [18; p 51]), we obtain for the flow rule

$$dE_\beta^{P\alpha} = \frac{\partial\phi}{\partial\sigma_\alpha^\beta} d\bar\lambda = 2 (\sigma_\beta^\alpha - \frac{1}{3} \delta_\beta^\alpha \sigma_\gamma^\gamma) d\bar\lambda \quad . \tag{C.10}$$

When we write the flow rule in a form more appropriate to our formulation using finite increments:

$$\Delta E_\beta^{P\alpha} = 2 (\sigma_\beta^\alpha - \frac{1}{3} \delta_\beta^\alpha \sigma_\gamma^\gamma) \Delta\bar\lambda \quad , \tag{C.11}$$

the question immediately arises as to what values of the stress components to use in this formula. Since the strain increments are associated with the time internal [t - Δt, t], ideally, values of the components of stress should be for some intermediate time, say $\sigma_\beta^\alpha(t - \frac{1}{2} \Delta t)$, in order that the stress be properly centered. However, such intermediate values are undetermined. Moreover, using the average

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of the values at t - Δt and t, vis. $\frac{1}{2} [\sigma_\beta^\alpha(t - \Delta t) + \sigma_\beta^\alpha(t)]$, would not only lead to computational difficulties since the values $\sigma_\beta^\alpha(t)$ are as yet unknown, but would be theoretically wrong since the average would not give a state of stress on the yield surface due to the strict con-- vexity of this surface. For these reasons we are persuaded to use the values at the time t - Δt for the stress components in (C.11). With this understanding, we substitute (C.11) in (C.5) to obtain

$$\Delta\sigma_\beta^\alpha = \frac{E}{1+\nu} [\Delta E_\beta^\alpha + \frac{\nu}{1-\nu} \delta_\beta^\alpha \Delta E_\gamma^\gamma$$

$$- 2 \{\sigma_\beta^\alpha(t - \Delta t) - \frac{1-2\nu}{3(1-\nu)} \delta_\beta^\alpha \sigma_\gamma^\gamma(t - \Delta t)\} \Delta\tilde{\lambda}] \quad , \quad (C.12)$$

or using the abbreviations

$$\overset{E}{\Delta\sigma}{}_\beta^\alpha = \frac{E}{1+\nu} [\Delta E_\beta^\alpha + \frac{\nu}{1-\nu} \delta_\beta^\alpha \Delta E_\gamma^\gamma] \quad ,$$

$$\overset{C}{\sigma}{}_\beta^\alpha = \sigma_\beta^\alpha - \frac{1-2\nu}{3(1-\nu)} \delta_\beta^\alpha \sigma_\gamma^\gamma \quad , \quad (C.13)$$

$$\Delta\lambda = \frac{2 E}{1+\nu} \Delta\tilde{\lambda} \quad ,$$

obtain the compact relation

$$\Delta\sigma_\beta^\alpha = \overset{E}{\Delta\sigma}{}_\beta^\alpha - \Delta\lambda \overset{C}{\sigma}{}_\beta^\alpha(t - \Delta t) \quad . \quad (C.14)$$

We now can proceed to determine the values of the stress components at the time t by substituting

$$\sigma_\beta^\alpha(t) = \sigma_\beta^\alpha(t - \Delta t) + \Delta\sigma_\beta^\alpha(\Delta\lambda) \quad (C.15)$$

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in the yield condition (C.2) resulting in a quadratic expression in $\Delta\lambda$ (notice the dependence of $\Delta\sigma_\beta^\alpha$ on $\Delta\lambda$ in (C.15), which we solve for the smallest positive value of $\Delta\lambda$ satisfying

$$\phi(\sigma_\beta^\alpha(t)) = 0 \quad . \tag{C.16}$$

This value of $\Delta\lambda$ will satisfy the quadratic equation

$$A \, \Delta\lambda^2 - 2B \, \Delta\lambda + C = 0 \quad , \tag{C.17}$$

where

$$A = \overset{C}{\sigma_\beta^\alpha}(t - \Delta t) \; \overset{C}{\sigma_\alpha^\beta}(t - \Delta t) - \frac{1}{3} [\overset{C}{\sigma_\alpha^\alpha}(t - \Delta t)]^2 \quad ,$$

$$B = \overset{T}{\sigma_\beta^\alpha}(t) \; \overset{C}{\sigma_\alpha^\beta}(t - \Delta t) - \frac{1}{3} \overset{T}{\sigma_\alpha^\alpha}(t) \; \overset{C}{\sigma_\beta^\beta}(t - \Delta t) \quad ,$$

$$\tag{C.18}$$

$$C = \phi(\overset{T}{\sigma_\beta^\alpha}(t)) \quad ,$$

$$\overset{T}{\sigma_\beta^\alpha}(t) = \sigma_\beta^\alpha(t - \Delta t) + \overset{E}{\Delta\sigma_\beta^\alpha} \quad .$$

Details of how the value of $\Delta\lambda$ is determined from (C.17) as well as a procedure for dealing with the case of complex $\Delta\lambda$ can be found in [4; Sect. IV].

We can summarize the procedure for determining the values of the stress components as follows: first a trial stress $\overset{T}{\sigma_\beta^\alpha}(t)$ is determined from the previous stress $\sigma_\beta^\alpha(t - \Delta t)$ and the strain increment using

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$(C.13)_1$ and $(C.18)_4$. If the trial stress $\overset{T}{\sigma}{}^{\alpha}_{\beta}(t)$ satisfies the yield condition:

$$\phi(\overset{T}{\sigma}{}^{\alpha}_{\beta}(t) \leqslant 0 \quad , \tag{C.19}$$

then the trial stress is the stress at the time t:

$$\sigma^{\alpha}_{\beta}(t) = \overset{T}{\sigma}{}^{\alpha}_{\beta}(t) \quad . \tag{C.20}$$

If the trial stress does not satisfy the yield condition:

$$\phi(\overset{T}{\sigma}{}^{\alpha}_{\beta}(t)) > 0 \quad , \tag{C.21}$$

then solving (C.17) for a proper value of $\Delta\lambda$, the stress at the time t is given by

$$\sigma^{\alpha}_{\beta}(t) = \sigma^{\alpha}_{\beta}(t) - \Delta\lambda \, \overset{C}{\sigma}{}^{\alpha}_{\beta}(t - \Delta t) \quad , \tag{C.22}$$

wherein $\overset{C}{\sigma}{}^{\alpha}_{\beta}(t - \Delta t)$ is determined from the stress at the time $t - \Delta t$ using $(C.13)_2$.

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APPENDIX D.   PHYSICAL COMPONENTS OF SURFACE STRAIN

In this appendix we derive the expressions used in the REPSIL code for the physical components of strain on the bounding surfaces of a shell. These expressions hold for a thin Kirchhoff shell and simulate the readings of strain gages bonded to the shell. They are exact with no restrictions as to the smallness of strain components being imposed in the derivation.

In Chapter 3 we derived expressions (3.2) and (3.3) for the current and initial differential arc lengths between neighboring shell particles. For particles on the bounding surfaces of the shell, $\zeta = \pm \frac{h}{2}$ and hence $d\zeta = 0$, so that these expressions become

$$ds^2 = g_{\alpha\beta} \, d\xi^\alpha \, d\xi^\beta \quad , \qquad dS^2 = G_{\alpha\beta} \, d\xi^\alpha \, d\xi^\beta \quad . \qquad (D.1)$$

The strain E that a fibre connecting neighboring particles on the surface undergoes is by definition

$$E = \frac{ds}{dS} - 1 \quad . \qquad (D.2)$$

This gives us

$$(E + 1)^2 = g_{\alpha\beta} \, \tau^\alpha \, \tau^\beta \quad , \qquad (D.3)$$

where $\tau^\alpha \equiv \dfrac{d\xi^\alpha}{dS}$ are the components of a unit vector $\underset{\sim}{\tau}$ in the initial basis, since clearly

$$G_{\alpha\beta} \, \tau^\alpha \, \tau^\beta = 1 \quad . \qquad (D.4)$$

Substituting definition (3.5) in (D.3) and taking account of (D.4), we relate the physical strain E to the tensor components of strain $E_{\alpha\beta}$:

$$(E + 1)^2 = 2E_{\alpha\beta} \, \tau^\alpha \tau^\beta + 1 \quad . \tag{D.5}$$

Since $\underset{\sim}{\tau}$ is a unit vector in the plane of the basis $\underset{\sim\alpha}{G}$, it is completely specified by the angle θ_1 it makes with $\underset{\sim 1}{G}$ or, equally well, the angle θ_2 it makes with $\underset{\sim 2}{G}$. Hence, we can write its components as follows

$$\tau^1 \equiv \underset{\sim}{G}^1 \cdot \underset{\sim}{\tau} = \sqrt{G^{11}} \sin \theta_2 \quad , \qquad \tau^2 \equiv \underset{\sim}{G}^2 \cdot \underset{\sim}{\tau} = \sqrt{G^{22}} \sin \theta_1 \quad . \tag{D.6}$$

Clearly, θ_1 and θ_2 are not independent, but are related through the metric $G_{\alpha\beta}$:

$$\cos (\theta_1 + \theta_2) = \frac{G_{12}}{\sqrt{G_{11} G_{22}}} \quad , \qquad \sin (\theta_1 + \theta_2) = \sqrt{\frac{G}{G_{11} G_{22}}} \quad , \tag{D.7}$$

with G the determinant of the metric:

$$G = G_{11} G_{22} - (G_{12})^2 \quad . \tag{D.8}$$

Using $(D.7)_2$, we can rewrite (D.6):

$$\tau^1 = \frac{1}{\sqrt{G_{11}}} \frac{\sin \theta_2}{\sin (\theta_1 + \theta_2)} \quad , \qquad \tau^2 = \frac{1}{\sqrt{G_{22}}} \frac{\sin \theta_1}{\sin (\theta_1 + \theta_2)} \quad . \tag{D.9}$$

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Substitution of the last expressions in (D.5) gives us the strain E as a function of the angle the fibre makes with the basis $\underset{\sim}{G}_\alpha$:

$$(E + 1)^2 = \frac{2}{\sin(\theta_1 + \theta_2)} [\epsilon_1 \sin^2 \theta_2$$

$$+ 2\gamma \sin \theta_1 \sin \theta_2 + \epsilon_2 \sin^2 \theta_1] + 1 , \qquad (D.10)$$

where we use the abbreviations

$$\epsilon_1 = \frac{E_{11}}{G_{11}} , \qquad \gamma = \frac{E_{12}}{\sqrt{G_{11} G_{12}}} , \qquad \epsilon_2 = \frac{E_{22}}{G_{22}} . \qquad (D.11)$$

Successively setting θ_1 and θ_2 equal to zero, we obtain the strain in the $\underset{\sim}{G}_1$ and $\underset{\sim}{G}_2$ directions:

$$_1 = \sqrt{1 + 2 \epsilon_1} - 1 , \qquad E_2 = \sqrt{1 + 2 \epsilon_2} - 1 . \qquad (D.12)$$

Since θ_1 and θ_2 are dependent, we may eliminate either one from (D.11) through the use of (D.7). We proceed to eliminate θ_2 by dividing the trigonometric identity

$$\sin \theta_2 = \sin(\theta_1 + \theta_2) \cos \theta_1 - \cos(\theta_1 + \theta_2) \sin \theta_1 \qquad (D.13)$$

by $\sin(\theta_1 + \theta_2)$ and using (D.7) with the abbreviations

$$\delta = \frac{G_{12}}{\sqrt{G_{11} G_{22}}} , \qquad \Delta = \frac{1}{\sqrt{1 - \delta^2}} , \qquad (D.14)$$

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to obtain

$$\beta(\theta_1, \delta) \equiv \frac{\sin \theta_2}{\sin (\theta_1 + \theta_2)} = \cos \theta_1 - \delta \Delta \sin \theta_1 . \qquad \text{(D.15)}$$

Using (D.7) and D.14) again, we also have

$$\alpha(\theta_1, \delta) \equiv \frac{\sin \theta_1}{\sin (\theta_1 + \theta_2)} = \Delta \sin \theta_1 . \qquad \text{(D.16)}$$

On substituting the last two expressions in (D.10) and solving for E, we obtain the strain for a fibre in the e_1 direction:

$$E = [2(\beta^2 \epsilon_1 + 2 \alpha \beta \gamma + \alpha^2 \epsilon_2) + 1]^{\frac{1}{2}} - 1 . \qquad \text{(D.17)}$$

Equations (D.12) and (D.17) are the ones used in REPSIL to compute the physical strains on the bounding surfaces of the shell.

Lastly, we show that when the basis is initially orthogonal, so that $G_{12} = 0$, and the strain components are small, ϵ_1, ϵ_2 and γ become the usual small strain components; for clearly we then have from (D.12) that

$$E_1 \approx \epsilon_1 , \qquad E_2 \approx \epsilon_2 . \qquad \text{(D.18)}$$

Also from (D.1) and (D.11)$_2$ we have

$$\gamma = \frac{g_{12}}{2\sqrt{G_{11} G_{22}}} = \frac{1}{2} \left(\frac{g_{11} g_{22}}{G_{11} G_{22}} \right)^{\frac{1}{2}} \cos (\frac{\pi}{2} - 2\Gamma) , \qquad \text{(D.19)}$$

whe.: 2Γ is the change the angle between the basis vectors experience is going from G_α to g_α. Using (D.11) and (D.12), the last expression

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becomes

$$\gamma = \frac{1}{2} (1 + E_1) (1 + E_2) \sin 2\Gamma , \qquad (D.20)$$

which for small strains is approximated by

$$\gamma \approx \Gamma . \qquad (D.21)$$

Hence, for small strains ε_1 and ε_2 approximate the strain undergone in the G_1 and G_2 direction while γ approximate half the angle change undergone between the directions G_1 and G_2; these are the usual small strain definitions for the strain components in the plane.